Students' Error in Derivatives of Functions Concept

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Abstract. This study aims to know students' errors in solving the problems in derivatives of functions concept. It is a preliminary research which using a qualitative method. It involved 27 students in 12th grade in senior high schools who have learned derivatives of functions. Data are collected from four test items in form of essay and questionnaires given to the students. The results showed that three types of students' error. There is the conceptual error where students could not understand the concept found in the problems, the procedural error where students failed to do algebraic manipulation, and the technical error caused by the lack of content knowledge on mathematics or caused by the carelessness. Another finding was that the students found it difficult to solve contextual problems.

INTRODUCTION

Hiebert and Lefevre classify mathematical knowledge into two kinds, namely "conceptual knowledge" and "procedural knowledge" [1]. The term "conceptual knowledge" is a rich knowledge in relationships, whereas "procedural knowledge" is divided into two parts. The first consists of formal languages or symbol representation in mathematics. The second part consists of algorithms or rules for completing mathematical tasks.

Another perspective was given by Harel. Harel defines mathematics as two parts completing each other, such as ways of understanding (WoU) and ways of thinking (WoT) [2]. WoU is a way resulting mathematical objects like axioms, definitions, theorems, proofs, problems, and solution. On the other hand, WoT is a way of thinking using particular characteristic of mental action to produce a set of mathematical objects. The perspective from Harel is illustrated by Suryadi as triadic model mental action-WoU-WoT implied towards the meaning of mathematics learning [3]. From the description, when a person learns mathematics particularly when solving mathematical problems, then they are doing a set of mental actions directed to the solution. The activities can create an image of someones' self continually. Indirectly, every mathematical problem presented affect the formation of someone's perception of a concept. This becomes an experience becoming one of the factors of the existence of some concept characteristics on someones' cognitive structure. The image of a concept planted to someone is affected by capacity, learning experience, and knowledge that someone has [3].

The "concept image" was introduced by Vinner through his theory of cognitive models. The theory illustrates that there are two different cells within a cognitive structure, i.e. definition cell and image cell [4]. An "image cell" is an image of a concept that describes the meaning of something in a person. The concept image can be known by giving a mathematical problem that has been studied previously. It is to know the meaningfulness of existing concepts in students. However, there are times when the concept of images planted in students is not in accordance with the scientific conception. One of the symptoms of the gap is the errors appears when solving mathematical problems.

Student's error is the symptom of misunderstanding [5]. It is inevitability found some errors when students solve mathematical problems. Like in the learning process, someone is never regardless of the error. Many studies focus on the analysis of error in learning mathematics such as the study of algebraic thinking, derivatives of trigonometric functions, Pythagoras theorem [6, 7, 8]. Both errors and success experienced by someone indirectly lead to success [9]. Besides, errors made can be a description of how someone has learned.

Donaldson classifies errors into three categories, that is the structural error, the arbitrary error, and the executive error [10]. The structural error is the error which appears from the failure appreciating the relation involved in a problem or understands some important principles for a solution. The arbitrary error is the error when students arbitrarily do everything their wants. The executive error is the error when students fail to manipulate, even though principles involved have been probably understood. On the other hand, Kiat develops Donaldsons' idea to classify error into three categories, that is conceptual error, the procedural error, and the technical error [11]. The operational definition proposed by Kiat is principally similar to the operational definition proposed by Donaldson. Kiat views conceptual error as structural error, procedural error as executive error, and technical error as arbitrary error.

Some research on students' understanding of derivative concepts has been widely studied by other researchers. Its findings show that some students' difficulties in the concept of derivative [7, 12]. However, these studies focus on the university level. We tried to reveal the root of the problem. We conducted an analysis of the education curriculum in Indonesia. We found that the derivative concepts were first learned in the 11th grade in senior high school. Therefore, we decided to focus on investigating the error profile of senior high school students on the concept of derivatives of functions.

The purpose of this paper was to know the error appear when students are given problems about the concept of derivatives of functions. This paper is drawn up referring to the research question that is "what kind of error appear when students solve the problems in derivatives of functions concept?".

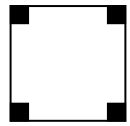
RESEARCH METHOD

This study is preliminary research which using a qualitative method. Data are collected from four test items in form of essay and questionnaires given to the students. The questions were used to identify the types of error that occurs when students solve the problems. The Types of questions were adapted from the national exam test for MIA program at the senior high school in recent years. "MIA" is a term for science majors in Indonesia. The test was given to the students who had studied the concept of derivatives of functions. The test was given to 27 students of 12th grades of MIA program at a senior high school in Bandung, Indonesia.

The arrangement of these four questions was based on the regulation of the minister of education and culture of Republic of Indonesia No.24, 2016 on core competence and basic competence of lessons in the 2013 curriculum on primary and secondary educations (description in Table 1) [13]. The questionnaire consists of 10 questions to know the students' opinions after solving the derivatives of functions problem. The question description includes students' legibility; students' understanding of the questions asked; students' experience of working on similar questions, student limitations in solving problems.

TABLE 1. Problems about derivatives of functions

Basic competence	Problems	Test item
Solving problems related to derivatives functions.	Find the derivatives of functions of $f(x)=x^2(2x+1)^3$!	1
Using the first derivative function to determine the maximum point, minimum point, and monotonic interval function, as well as the slope of the curve's tangent, the equation tangent, and the normal line of the curve	Given the function curve $y=x^2-4x+5$ intersects the line $y = x+1$. Determine the equation of the tangent through the curve point and the line!	2
	Given function $g(x)=1/3\ x^3-A^2x+7$, A is constant. If $f(x)=g(2x+1)$ and f decrease at $-3/2 \leq x \leq 1/2$, then find the relative minimum value of g!	3
	A square carton has an area of 30 centimeters squared. It will be made a box without a cover by cutting four boxes on each corner of the box (as in the following picture). a. Specify length, width, and height for maximum volume of the box!	4

corresponding to the contextual problem.	b. What is the maximum volume of boxes that have been established? 

RESULTS AND DISCUSSION

Based on the classification of the errors proposed by Donaldson and Kiat, we classify the types of the errors that appear into three categories, namely conceptual error, procedural error, and technical error. The operational definition of these three types of errors refers to opinion proposed by Kiat that conceptual error refers to errors due to failure to understand concepts involved in problems or errors arising from failure to appreciate the relationships involved in the problems; procedural error are errors arising from failure to manipulate or algorithm despite having understood the concept behind the problem; and technical error is an error due to lack of knowledge of mathematical content or due to carelessness. In Table 2 we present a summary of the results of the analysis of the student responses and the error categories appear.

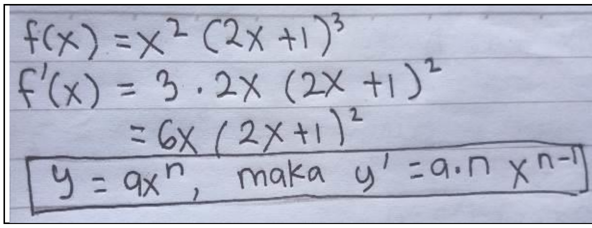
TABLE 2. The summary of the students' responses and error categories on each question

Item	Students' responses	Error categories	Students' number
1	Use the theorem of multiplication, if $F(x)=f(x).g(x)$ so $F'(x)=f'(x).g(x)+f(x).g'(x)$. However, wrong in applying the formula	Procedural error, technical error	7
	Use the theorem, if $f(x)=ax^n$ so $f'(x)=anx^{n-1}$	Conceptual error, technical error	5
	Giving complete explanation and correct answers	No errors	15
2	No response	Difficult to understand the question	2
	Derivatives $y=x^2-4x+5$ is $y'=2x-4$ and $m=2$	Conceptual error, procedural error	4
	Suppose that the cross line of parabola and line is on $x^2-4x+5=0$, then seeking the equation of normal line	Conceptual error, procedural error	1
	The intersection of parabola and the line is $x^2-5x+6=0$. Errors in manipulating the algebraic	Procedural error, technical error	3
	Complete finishing and correct answer	No error	17
3	No response	Difficulty in understanding the question	10
	There is response but doing error in understanding the minimum concept	Conceptual error	4
	Complete finishing and correct answer	No error	13
4	No response	Difficulty in understanding the question	12
	Wrong understanding of complete finishing	Conceptual error	6
	Complete finishing but there is still minor error	Technical error	8
	Complete finishing and correct answer	No error	1

1. Analysis problem 1

Some types of error illustrated through the students' responses in solving the test number one. The first response was the students solved question number one using the multiplication theorem on the derivatives of functions. Student gave $x^2=u$ and $(2x+1)^3=v$. After that, the first derivatives of functions was able to be searched by the $f'(x)=u'v+v'u$. However, when the students took $(2x+1)^3=v$, they made the mistake in finding the first derivatives of $v'=3(2)^2=12$ or $v'=3(2x+1)^2$. In another response, it was also found that the students answered $v'=3(2x+1)^2=(6x+3)^2$. The students considered that the form was equivalent to $x(a+b)=xa+xb$. In other cases, there were students who gave $2x+1=v$ and found the first derivation of v that $v'=(6x)^2$. These findings suggest that students are lack in algebraic manipulations [14].

The second response was the student wrote on the answer sheet that the problem became $f(x)=x^2(2x+1)$. The students then used the formula if $f(x)=ax^n$ then $f'(x)=anx^{n-1}$. The answer was $f'(x)=2x^3+x^2=6x^2+2x$. through the answer, we assume that for the form was not the derivatives of functions of $f(x)$, but rather the conversion of the multiplication distributive property in addition. On the other hand, we assume that the students do not really understand the multiplication derivatives properties. The other response appears that students considered the first derivative of functions $f(x)=x^2(2x+1)^3$ is $f'(x)=3.2x(2x+1)^2$ (see Figure 1). The illustration of the answer implicitly demonstrates that students understand the derivatives partially. Furthermore, the students do not look at the form $(2x+1)^3$ as a function that can be found for its derivatives of functions using chain rules or through conversion.



Translation:
$f(x) = x^2(2x+1)^3$ $f'(x) = 3.2x(2x+1)^2$ $= 6x(2x+1)^2$ $y = ax^n$, so $y' = a.nx^{n-1}$

FIGURE 1. Example of students' response on problem 1

Through both responses, we categories the types of errors into conceptual errors, procedural errors, and technical errors. The conceptual errors occur because the students do not understand the derivative principles of the multiplications' product and do not understand the derivation properties, particularly on the chain rules' properties. Another type of error is procedural errors. It occurs when the students fail at every step of doing the algebraic manipulation. The failure can be caused by lack of understanding or students can understand the theory but cannot apply it to solving the problem. Accuracy is possibly necessary when solving mathematical problems. If the students are not careful in understanding and paying attention to test, they may be solving the problem carelessly. Such conditions may lead to other types of errors i.e. technical errors.

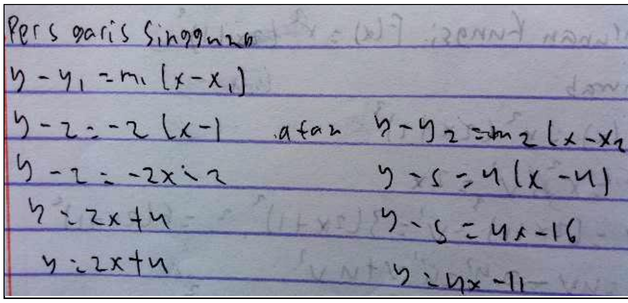
2. Analysis problem 2

The first type of response found was on completing the derivation of $y=x^2-4x+5$ is $y'=2x-4$. The students wrote that the value of $m=2$. From the response, we assume that the students considered the first derivative of the function $y'=2x-4$ equal to the form of $y=mx+c$. The students interpreted the gradient value as a value attached to variable x. It affects the next mental activity chosen by the students. In the next mental activity, the students had correctly applied the equation formula of the tangent line through the point (x_0,y_0) i.e $y-y_0=m(x-x_0)$. Nevertheless, the misinterpretation in understanding the value of the gradient leads to continuous errors in the solution achieved.

The second type of response was the students' misinterpretation in finding intersections between the curves $y=x^2-4x+5$ with the line $y=x+1$. The students sought the intersection by factoring x^2-4x+5 to $(x-5)(x-1)$ and substituting the value of x to the equation curve. At another stage, the students looked for a gradient with the same procedure on the first response type that assumed the coefficient on variable x was the gradient value of m_1. It does not end here. The students considered that the tangent equation in question was a normal line equation. The students then looked for the gradient of the tangent with the concept of $m_1.m_2=-1$and solved the tangent equation.

The third type of response was some error finding in algebraic manipulation even though the students understand the concept of the problem presented. Like the first example of student failure in completing the form of $x^2-4x+5=x+1$. The result of algebraic manipulation found by the students was $x^2-5x+6=0$. The second example is when the student got the form of $(y-2)=-2(x-1$ manipulated into $y-2=-2x-2$ (see Figure 2). However, in the next stage, the students answered $y=2x+4$. The third example is when the student found the solution for $(y-2)= -2(x-1)$ and manipulated into $y=-2x+4$ /$2x-4$.

Through both responses, we categorize the types of error as conceptual errors, procedural errors, and technical errors. It is a conceptual error when the students do not understand the tangent gradient, the relationship between the tangent gradient and the first derivation. The second type of error is a procedural error caused by the failure of manipulation even though they had understood the concept of the problem. Meanwhile, technical errors occur due to students' inattentiveness when solving the problem.



Translation:

Equations of tangent

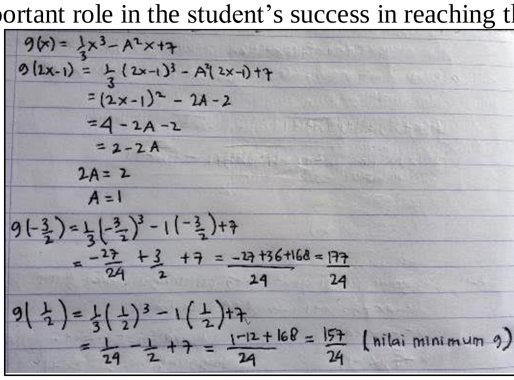
$y-y_1= m_1 (x-x_1)$		
$y-2= -2(x-1$	or	$y-y_2=m_2(x-x_2$
$y-2=-2x-2$		$y-5=4(x-4)$
$y=2x+4$		$y-5=4x-16$
$y=2x+4$		$y=4x-11$

FIGURE 2. Example of students' response on problem 2

3. Analysis problem 3

The first type of the students' response occurred when completing the first derivative of $f(x)$. In Figure 3, it appears that the students directly solved the first derivative of $g(2x+1)$ without trying to decipher the form of $f(x)=g(2x+1)$. The steps taken might lead to potential errors. In addition, the students considered the value of A as a variable, so that the students looked for a derivative of the function of A. The students also do not understand the concept of a minimum relative. It can be seen when the students substituted the value of $-3/2$ and $1/2$ to the function $g(x)$.

The students' error in problem 3 is a conceptual error, in which the students failed to understand the concept of the problem. Students preference for procedural methods rather than conceptual understanding [14]. The various concepts include the concept of a minimum relative concept. The background knowledge of the function plays an important role in the student's success in reaching the solution of the problem.



Translation:

$$g(x) = \tfrac{1}{3}x^3 - A^2x + 7$$

$$\begin{aligned} g(2x-1) &= \tfrac{1}{3}(2x-1)^3 - A^2(2x-1) + 7 \\ &= (2x-1)^2 - 2A - 2 \\ &= 4 - 2A - 2 \\ &= 2 - 2A \\ 2A &= 2 \\ A &= 1 \end{aligned}$$

$$g\left(-\tfrac{3}{2}\right) = \tfrac{1}{3}\left(-\tfrac{3}{2}\right)^3 - 1\left(-\tfrac{3}{2}\right) + 7 = -\tfrac{27}{24} + \tfrac{3}{2} + 7 = \tfrac{-27+36+168}{24} = \tfrac{177}{24}$$

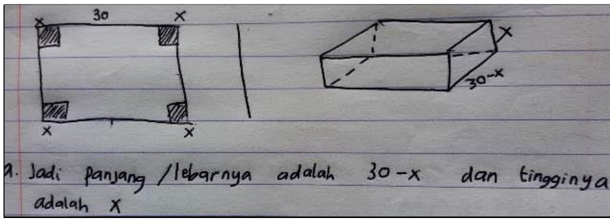
$$g\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\left(\tfrac{1}{2}\right)^3 - 1\left(\tfrac{1}{2}\right) + 7 = \tfrac{1}{24} - \tfrac{1}{2} + 7 = \tfrac{1-12+168}{24} = \tfrac{157}{24} \text{ (the minimum value of } g)$$

FIGURE 3. Example of students' response on problem 3

4. Analysis problem 4

The first type of the students' response when solving the problem was the students' mistake in modeling the problem from the real form into the mathematical model. The students answered point a that "so, the length or the width is $30-x$ with height of x". The answer comes with a boxless illustration box that has a length or width of $30-x$ and a height of x (see Figure 4).

	Translation:



a. So the length is $30 - x$, the width is $30 - x$, and the hight is x

FIGURE 4. Example of students' response on problem 4

The second type of response, the students modeled the complete box with the length or width of 30−*x* and the height of *x*. However, the concern is that when the students solved the question on point b, they wrote "*V*=surface area multiply height=$(30-2x)^2.x$" and were able to solve it correctly, although the answer is indirectly contradictory to the answer in point a that the length or width of the box is 30−*x*.

From the analysis of the results showed that conceptual error, procedural error, and technical error are related to each other. If someone does not understand a concept correctly then it will have an opportunity to display errors when troubleshooting. Furthermore, a concept that is correctly understood not necessarily can be communicated through the language of mathematics correctly. Although the solution is correct, sometimes someone is not careful about the idea of carelessness. This finding suggests that the importance of developing conceptual knowledge and procedural knowledge. Another interesting finding is that most students fail to solve contextual problems. Tall notes that when a person is confronted with a contextual problem, they will translate into an abstract form, from an abstract problem then solved, and the solution obtained is reinterpreted in contextual matters [15]. From Talls' view and the findings on number 4 denoted that most students fail when translating from real problem to abstract form. It because students have difficulties in translating real-world problems into calculus formulation [14].

The results of the questionnaire indicated that most students found it difficult to solve the problems. They did not understand the questions presented. A total of 11 students did not understand the problem on number 3, and 12 students did not understand the problem on number 4. In addition that 15 students found it difficult when they were asked to converted the problem into the mathematical model. A total of 25 students found it difficult to relate the concept one another. The result related to the students' errors that appeared in solving the problem of the derivatives of functions. On the other hand, students who do not find it difficult to solve problems but display errors both conceptually, procedurally, and technically. We guess that students chose the mistakes as their truths. It condition illustrates the gap between the concepts embedded in the mind of the students with the formal concept.

CONCLUSION

Through the description of student responses that appear from items 1 to 4, it is found that most students experience errors when working on derivatives of functions problems. In outline, we categorize into three types of students' errors, namely conceptual error as a result of the failure of students in understanding the concept involved in the problem of derivatives of functions.The second error is that the procedural error contains errors in algebraic manipulation. The third error is a technical error due to a lack of mathematical content knowledge mainly on prerequisite knowledge. The technical error also emerged as a result of carelessness done by students. Another type of error that the author gained is an error in solving contextual problems.

This study obtained information that students are more focused on procedural knowledge to solve a problem. Students prefer to interact with mathematical symbols. Students have not focused on the conceptual knowledge rich in relationships. Star believes that conceptual knowledge is a well-known knowledge, whereas procedural knowledge is a superficial knowledge [16]. However, establishing connections between procedural knowledge and conceptual knowledge is equally important in understanding the concept correctly and in depth [17].

Hopefully, this paper can be a contribution to the environment of mathematics education. However, because of the limited time and opportunities owned by the authors so this paper is still a lot of deficiencies. This causes many questions in our minds that may be a source of problems for further research. Finally, we get some fundamental questions that are: (1) what causes the emergence of errors or difficulties in the concept of derivatives of functions, (2) what the meaning of derivative according to students, (3) how the experience of meaning experienced by students on the concept of derivatives of functions.

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