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Mathematical Generalization : A Systematic Review and Synthesis of Literature

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Abstract. This paper provides an analysis of the existing literature on the mathematical generalization focusing on empirical research findings. Out of all searched citations, a total of 5 publication from mathematics education met the inclusion criteria. The analysis was based on the types of generalizations' category (process and product), topic (material of mathematical learning), and some indicator to determine mathematical generalization of students. The generalization process can be known by looking at the students in relating, searching relationship, and extending two or more problems, situations, ideas, or mathematical objects. The generalization product indicates specific indicator of the mathematical generalization abilities. Indicators of mathematical generalization ability are grouped into three main categories, namely identification or statement, definition, and influence. The aspect of identification or statement, students can identify or express the properties, similarities or general principles of rules, patterns, strategies or procedures, and general rules of an object, situation or phenomenon. The aspect of the definition, students can define an object class that satisfies a given relationship, pattern, or phenomenon. The aspect of influence, students can apply or modify general ideas or strategies to solve situations or problems in mathematics. In addition, these findings provide insights into certain challenges and confounding factors involved in designing new assessment instruments for each indicator of the mathematical generalization abilities. And mathematical generalization abilities of students cannot be seen only in certain material, but more broadly such as numerical, algebraic, and statistic.

INTRODUCTION

Generalization, a process of deducing certain things into the form of a general statement or concept, has received attention as one of the focuses of curriculum development of school mathematics [1]. Mathematical generalization has been widely recognized as a challenge for many students [2,3,4,5,6]. Students can be adept at making all kinds of generalizations, but this does not mean that they will be able to generalize in ways that are productive in terms of being mathematically useful, or helpful in achieving the mathematical goals of a lesson or a unit [7,8,4]. Not only is important in school learning, students' mathematical generalization abilities are also useful in real life.

In learning, as a social-cognitive process, generalizations can train students to be more communicative. This is because in the individual cognitive occurs a psychological process, and generalization is one of them. Such generalization would result in a cognitive construction product which processes are always conditioned and socially mediated for using and relying on the means provided and prepared by society such as language [9]. In addition, students also become more creative in solving problems or math in a real context by combining their knowledge and experience [10]. Without generalization, there is no way for teachers and students to cope with an unlimited amount of information in both the abstract and the real world [11].

Generalizations in mathematics are heartbeat and life-blood. A mathematical thinking in learning occurs only when the teachers aware of generalization form and students inhabit to express their generalizations. In other words, as an intrinsic element for activity and thinking, generalization is what makes it mathematical [12, 13, 14].

[15] argues that mathematical generalization has to do with noticing patterns and properties common to several situations. Mathematical generalization is not only the basic and characteristics way of thinking and reasoning to indicate the process by which concepts are seen in the wider context, but also the product of the process [16,17]. The first step in the generalizing process is the formation of a clear concept. First, students give the idea of

"common attributes" to a group of specific objects. Then, by differentiating attributes, students will assign labels or categories to group members. This labeling or category is used to differentiate and find similarities between concepts. If the students can see the relationship of a concept with another concept in a particular situation, then they have reached the level of generalization [18]. The product which is the level of generalization is certainly related to the possession of student's mathematical generalization abilities.

Generalization ability defined by [19] as an ability to grasp what is common in different problems and examples and, correspondingly, to see what is different in the common begins to take shape earlier than all other components. This is in line with the [20] that defines that mathematical generalization ability is an ability to relay the patterns, determine the structure, data, images, or the next terms, and formulate a general symbolic. Thus, the ability of mathematical generalization is the ability of students to relay the patterns, determine the structure, data, images, or the form, and formulate a general symbolic from problem and example that vary.

The attempt to design the instruction to improve mathematical generalization ability had relied on a theoretical framework about the mathematical generalization, which analyzes the constituent components into two broad categories, which is generalizing actions and reflection generalization. Generalizing action is a process that could direct to improve student's generalization. Meanwhile, reflection generalization represents a student's ability to identify or use existing generalizations [21].

Generalizing actions consists of three major categories, they are relating, searching, and extending. In relating, students connect two or more problems, situations, ideas, or mathematical objects. In searching, students look for relationships, patterns, and solutions or similar results. In searching, students look for relationships, patterns, and solutions or similar results. In extending, students extend the pattern, relationship, or rule into a more mathematics' general structure. Then, reflection generalizations can be seen through several aspects, they are identification or statement, definition, and influence. In the aspect of identification or statement, students can identify or express the properties, similarities or general principles of rules, patterns, strategies or procedures, and general rules of an object, situation or phenomenon. In the aspect of the definition, students can define an object class that satisfies a given relationship, pattern, or phenomenon. Finally, in the aspect of influence, students can apply or modify general ideas or strategies to solve situations or problems in mathematics [21, 22]. (fig 1).

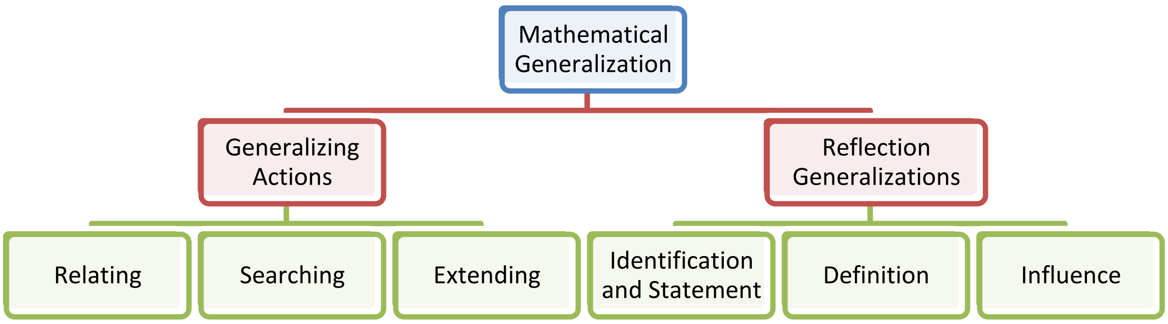

Fig 1. Mathematical Generalization Framework

In developing mathematical generalization ability, attempts are made to investigate the level of students' ability to generalize [23]. [12] describe that students who have generalization ability can be seen at least through three questions as follows: 'what seems likely to be true (a conjecture), why it is likely to be true (a justification), and where it is likely to be true, that is a more general setting of the question. Meanwhile, according to [20] which formulate five indicators to determine students' mathematical generalization ability, they are to recognize a rule/pattern, to describe a rule or pattern in numerical or verbal, to produce a general rule and pattern, to apply rule or pattern from various problems, and to formulate a general symbolic.

The fragmented view of how to know students' mathematical generalization and the need for generalization implementation in learning, highlights the need for a systematic review of mathematical generalization. This review study aims to contribute to process and reflection generalization in mathematics learning by systematically compiling the available empirical evidence on the various approaches as published in journal articles. In particular, this paper answers the following research question: how to know students' mathematical generalization abilities?

METHODS

This review paper followed [24] methodological guidelines for conducting a systematic literature review. We chose this method because the steps are clear enough to classify the data. Some of the main steps for conducting the literature search are as follows:

Constructing Search Terms

The following details will help in determining the search terms of our research question. Mathematical generalization: generalization, mathematical generalization, generalizing process, generalizing action, reflection generalization, mathematical generalization ability. The question containing the above detail is: how to know the ability of student's mathematical generalization?

Search Strategy

A search of international peer-reviewed published literature was undertaken with a focus on empirical studies that emphasized mathematical generalization. We searched keywords, titles and abstracts in four databases using the following keyword combinations: (generalization OR mathematical generalization) AND (student). Databases that used to search and filter out the relevant papers are JSTOR, Springer, Scopus, and SAGE.

Publication Selection

The study selection process is based on several criteria. This study selected papers that:

- related to the search term used, such as generalization, mathematical generalization, generalizing proccess, generalizing action, reflection generalization, mathematical generalization ability.;
- scope in mathematics education;
- used quantitative or qualitative empirical measures;
- described research involving students of any age in any educational system, who attended private or public schools;
- were written in English;

Coding Study Characteristics

Characteristics central to review and analyzed the articles selected on basis as follows:

- *Name of study*, the first author and the year of publication are stated
- *Category of generalization*, consist of generalizations' process and product
- *Topic*, learning materials used in empirical research
- *Indicator*, used of point view to determine the student's mathematical generalization

RESULTS AND DISCUSSIONS

After screening all titles of searching, this study assessing 33 abstract and full texts for eligibility, and then identified 5 publications that met the inclusion criteria. This can be seen in the following table.

TABLE 1 Data Sources and Results for Literature Search

Data Source	Primary Selection	Final Selection
Springer	16	1
SAGE	3	1
Scopus	2	0
JSTOR	12	3

Then, 5 data publications that have been analyzed, summarized in the table as follows:

TABLE 2 Summary of Studies in Term of Category, Topic, and Indicator in Mathematical Generalization

Reference	Category of Generalization	Topic	Indicator
[24]	Product	Statistic	(a) the ability to write a mathematical illustration exemplifying the same relationship as in several given illustrations of the relationship (b) the ability to write a word statement of the general truth or fact exemplified by several given illustrations of a mathematical relationship (c) the ability to illustrate a word statement of a general truth or fact in mathematics by writing a mathematical relationship
[14]	Proccess	Algebra	(a) identifying commonality across cases, (b) extending one's reasoning beyond the range in which it originated, (c) deriving broader results from particular cases
[25]	Proccess	Numeric	(a) *counting*, drawing a picture or constructing a model to represent the situation and counting the desired attribute (b) *recursion*, building on a previous term or terms in the sequence to construct the next term (c) *whole-object*, using a portion as a unit to onstruct a larger unit using multiples of the unit. This strategy may or may not require an adjustment for over-or undercounting (d) *contextual*, constructing a rule on the basis of a relationship that is determined from the problem situation (e) *guess and check*, guessing a rule without regard to why the rule may work (f) *rate-adjust*, using the constant rate of change as a multiplying factor. An adjustment is then made by adding or substracting a constant to attain a particular value of dependent variable
[26]	Proccess	Algebra	(a) grasping a commonality noticed on some particulars (b) extending or generalizing this commonality to all subsequent terms (c) being able to use the commonality to provide a direct expression of any term of the sequence
[27]	Proccess	Algebra	(a) *counting from a drawing* by which the students count the elements of a particular figural term in a pattern (b) *recursive* where the students points out the common difference between pairs of consecutive terms and use it to repetitively add from term to term to extend the pattern (c) *chunking* that involves the students in multiplying common difference between pairs of consecutive terms in the pattern by the number of steps and in adding the result to an initial figural term (d) *functional* that is characterized by relating parts of the pattern to the figural step number (e) *whole-object* that involves identifying the value of a term by using multiples of a previous term or by adding two previous terms

Based on the results of the research in Table 2, The mathematical generalization of students does not depend on a particular material. The material that can be used to view generalizations of student mathematics is broad enough, including numeric, algebra, and statistic. The results also show that there are some indicators that can be used to find out the mathematical generalization of student both in the process (generalizing actions) and products (reflection generalizations) [21, 22]. That details of the indicator can be seen in in the table as follows:

TABLE 3 Classification Indicator of Mathematical Generalization

Reference	Indicator	Major Category [21, 22]	Category of Generalization [21,22]
[25]	(a) the ability to write a mathematical illustration exemplifying the same relationship as in several given illustrations of the relationship	identification or statement	product
	(b) the ability to write a word statement of the general truth or fact exemplified by several given illustrations of a mathematical relationship	definition	
	(c) the ability to illustrate a word statement of a general truth or fact in mathematics by writing a mathematical relationship	definition	
[14]	(a) identifying commonality across cases,	searching	proccess
	(b) extending one's reasoning beyond the range in which it originated,	extending	
	(c) deriving broader results from particular cases	extending	
[26]	(a) *counting*, drawing a picture or constructing a model to represent the situation and counting the desired attribute	influence	product
	(b) *recursion*, building on a previous term or terms in the sequence to construct the next term	influence	
	(c) *whole-object*, using a portion as a unit to construct a larger unit using multiples of the unit. This strategy may or may not require an adjustment for over-or undercounting	definition	
	(d) *contextual*, constructing a rule on the basis of a relationship that is determined from the problem situation	definition	
	(e) *guess and check*, guessing a rule without regard to why the rule may work	influence	
	(f) *rate-adjust*, using the constant rate of change as a multiplying factor. An adjustment is then made by adding or substracting a constant to attain a particular value of dependent variable	influence	
[27]	(a) grasping a commonality noticed on some particulars	identification	product
	(b) extending or generalizing this commonality to all subsequent terms	definition	
	(c) being able to use the commonality to provide a direct expression of any term of the sequence	influence	
[28]	(a) *counting from a drawing* by which the students count the elements of a particular figural term in a pattern	influence	product

(b) *recursive* where the students points out the common difference between pairs of consecutive terms and use it to repetitively add from term to term to extend the pattern	influence
(c) *chunking* that involves the students in multiplying common difference between pairs of consecutive terms in the pattern by the number of steps and in adding the result to an initial figural term	influence
(d) *functional* that is characterized by relating parts of the pattern to the figural step number	identification
(e) *whole-object* that involves identifying the value of a term by using multiples of a previous term or by adding two previous terms	definition

Based on the mathematical generalization indicator classification in Table 3, the generalization process can be known by looking at the students in relating, searching relationship, and extending two or more problems, situations, ideas, or mathematical objects. The generalization product indicates specific indicator of the mathematical generalization abilities. Results of the generalization process as indicators of mathematical generalization ability are grouped into three main categories, namely identification or statement, definition, and influence. The aspect of identification or statement, students can identify or express the properties, similarities or general principles of rules, patterns, strategies or procedures, and general rules of an object, situation or phenomenon. The aspect of the definition, students can define an object class that satisfies a given relationship, pattern, or phenomenon. The aspect of influence, students can apply or modify general ideas or strategies to solve situations or problems in mathematics.

CONCLUSIONS

This paper has presented over seven decade's systematic review on mathematical generalization. This paper has also outlined several future insights on educational data clustering based on the existing literature reviewed, and further avenues for further research are identified. In summary, mathematical generalization abilities of student can be seen on many learning materials such as numeric, algebra, and statistic. Then, for knowing the mathematical generalization abilities of students can be done in two ways, namely by looking at the generalization process of students and see the results of the generalization process or mathematical generalization ability.

REFERENCES

1. NTCM, *Principles and Standards for School Mathematics* (NTCM, Inc., Reston, 2000), pp. 262.
2. J. Ainley, L. Bills, and K. Wilson, *International Journal of Computers for Mathematical Learning*, **10**, 191-215, (2005).
3. Lee. L, "An initiation into algebraic culture through generalization activities," in *Approaches to Algebra: Perspectives for Research and Teaching*, edited by N. Bednarz, C. Kieran and L. Lee (Kluwer Academic Publishers, Dordrecht, 1996), pp. 87-106.
4. K. Stacey, *Educational Studies in Mathematics*, **20**, 147-164, (1989).
5. K. Stacey and M. Mac Gregor, "Curriculum reform and approches to algebra," in *Perspectives on School Algebra*, edited by Sutherland, T. Rojano, A. Bell, and R. Lins (Kluwer Academic Publishers, Dordrecht, 2001), pp. 141-154.
6. J.R. Becker and F. Rineva, "Generalization strategies of beginning high school algebra students,". in *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education 4*, edited by H. L. Chick and J. L. Vincent (PME, Melbourne, 2005), pp. 121-128.
7. L. Lee and D. Wheeler, *Algebraic thinking in high school students: Their conceptions of generalization and justification* (Concordia University, Department of Mathematics, Montreal, 1987).
8. A. Orton and J. Orton, "Students' perception and use of pattern and generalization," in *Proceedings of the 18th Conference of the International Group for the Psychology of Mathematics Education 3*, edited by J. P. da Ponte and J. F. Matos (PME Program Committee, Lisbon, 1994), pp. 407–414.

9. W. Dörfler, "Forms and means of generalization in mathematics," in *Mathematical Knowledge: Its Growth Through Teaching*, edited by A. Bishop, et al (Kluwer Academic Publishers, Dordrecht, 1991), pp. 63-85.
10. N. Hashemi, M.S. Abu, H. Kashefi, and K. Rahimi, "Generalization in the Learning of Mathematics," in 2nd International Seminar on Quality and Affordable Education, (ISQAE, 2013), pp. 208–215.
11. P.A. Alexander and M.M. Duehl, "Seeing the possibilities: constructing and validating measures of mathematical and analogical reasoning for young children," in *Mathematical and Analogical Reasoning of Young Learners*, edited by L. English (Lawrence Erlbaum Associates Publishers, Mahwah, 2004), pp. 36.
12. J. Mason, L. Burton, and K. Stacey, *Thinking mathematically 2nd Edition* (Pearson, Harlow, 2010), pp. 8-22.
13. J. Mason, "Expressing generality and roots of algebra," in *Approaches to algebra: Perspectives for Research and Teaching*, edited by N. Bednarz, C. Kieran, and L. Lee (Kluwer Academic Publishers, Dordrecht, 1996), pp. 65–86.
14. J.J. Kaput, "Teaching and learning a new algebra," in *Mathematics Classroom that Promote Understanding,* edited by E. Fennema and T. A. Romberg (Lawrence Erlbaum Associates Inc, Mahwah, 1999), pp. 133–155.
15. J. Mason, *Learning and Doing Mathematics* (Macmillan Education Ltd, London, 1988), pp. 9-10.
16. D. Haylock and F. Thangata, *Key Concepts in Teaching Primary Mathematics* (SAGE Publication Ltd, London, 2007), pp. 79
17. D. Tall, *Advanced Mathematical Thinking* (Kluwer Academic Publishers, New York, 2002), pp. 11
18. M.A. John, *The Journal of Education*, **152**, No. 4, pp. 18-27, (1970).
19. V.A. Krutetskii, "*The Psychology of Mathematical Abilities in School Children*," (The University of Chicago Press, Chicago, 1976), pp. 334.
20. K. E. Lestari and M.R. Yudhanegara. *Penelitian Pendidikan Matematika* (PT Refika Aditama, Bandung, 2017), pp. 89.
21. A. B. Ellis, *Journal for Reseach in Mathematics Education*, **38**, No 3, pp. 194-229, (2007).
22. A. B. Ellis, *The Journal of Learning Sciences*, **16**, No 2, pp. 221-262, (2007).
23. J.A. Garcia-Cruz and A. Martinon, "Level of generalization in linear pattern," in *Proceeding of the 22nd Conference of the International Group for the Psychology of Mathematics Education 2*, (University of Stellenbosh, 1998), pp. 329-336.
24. B.A. Kitchenham, *Procedures for Undertaking Systematic Reviews, Joint Technical Report* (Keele University and National ICT Australia Ltd., 2004).
25. R.S. Ebert, *The Journal of Educational Research*, **39**, No. 9, pp. 671-681, (1946).
26. J.K. Lannin, *Mathematics Teaching in the Middle School*, **8**, No. 7, pp. 342- 348, (2003).
27. L. Radford, *ZDM*, **40**, No. 1, pp. 83-96, (2008).
28. M.E. Jurdak and R.R. El Mouhayar, *Educational Studies in Mathematics*, **85**, No. 1, pp. 75-92, (2014).