Parameter Estimation and Hypothesis Testing on Bivariate Generalized Poisson Regression

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Abstract—Poisson regression is regression method used to analyze response variable which is discrete. Equality of mean and variance (equidispersion) are the assumption that must be fulfilled in this model. If assumption is violated, the conclusion would be not valid. Wrong assumption occurs if variance greater than mean and is often called (overdispersion). But if variance less than mean it is called (underdispersion). There is no data used with excessive zero value on the response variable, therefore this research uses Bivariate Generalized Poisson Regression. Parameter estimation of Bivariate Generalized Poisson Regression is done by using Maximum Likelihood Estimation (MLE).

Keywords: overdispersion, underdispersion, bivariate generalized poisson regression, MLE

I. INTRODUCTION

Regression analysis is statistical method which is often used in science. The aim of analysis is modeling the relationship between two variables, which consist of predictor variable and response variable. In general, regression analysis is used in response variable which is continuous, but sometime response variable is discrete. Type of response variable is discrete which count data is non-negative that declare a lot of events in interval time, space or void volume. Data from an event would follow the Poisson distribution if such events are rare in a large sample[1]. This modeling is called Poisson Regression. There is assumption that must be met, mean and variance of response variable should be the same [6]. If assumption is violated, the conclusion would be not valid. Assuming violation occurs if variance is greater than mean is called overdispersion and if variance is less than mean is called underdispersion. Bivariate Poisson regression is used in data that have two response variables in data count with a high correlation.

II. LITERATURE

A. Bivariate Poisson Regression

Random variables $Y_1, Y_2$ is jointly bivariate Poisson distribution and according to [5] joint probability function:

$$f(y_1, y_2) = \left\{ \begin{array}{l} e^{-(\mu_1 + \mu_2)} \sum_{k=0}^{\min(y_1, y_2)} \frac{\mu_1^k \mu_2^{y_1-k} e^{-\mu_1}}{k!} \frac{\mu_2^k \mu_1^{y_2-k} e^{-\mu_2}}{k!} \mathbf{1}(y_1, y_2) = 0,1,2,..., (y_1, y_2) \text{ yang lain} \\ 0 \end{array} \right. $$

(1)

Bivariate poisson regression model can be written

$$(Y_1, Y_2) \sim PB(\mu_1, \mu_2, \mu_0)$$

$$\mu_j + \mu_0 = e^{\beta_j}, j = 1,2$$

(2)
Estimation method is used on Bivariate Poisson Regression is Maximum Likelihood Estimation (MLE) [4]. The method for calculating statistic test in parameter test is using Maximum Likelihood Ratio Test (MLRT):

\[
D(\hat{\beta}) = -2 \ln \left( \frac{L(\hat{\phi})}{L(\hat{\theta})} \right) = 2 \left( \ln L(\hat{\theta}) - \ln L(\hat{\phi}) \right)
\]

(3)

where

\(L(\hat{\theta})\) : Maximum Likelihood for complete model with predictor variable

\(L(\hat{\phi})\) : Maximum Likelihood for simple model without predictor variable

B. Generalized Poisson Regression

Generalized Poisson Regression is used to modeling overdispersion or underdispersion [2]. According to [3], Generalized Poisson Distribution has probability function:

\[
f(y; \mu, \alpha) = \left( \frac{\mu}{1 + \alpha \mu} \right)^y \frac{(1 + \alpha y)^{-1}}{y!} \exp \left( \frac{-\mu(1 + \alpha y)}{1 + \alpha \mu} \right)
\]

\(\mu = \mu(x) = \exp(x \beta)\)

(4)

Generalized Poisson Regression model have same form as Poisson Regression model:

\[
\mu_i = \exp(x_i^T \beta) = \exp(\beta_0 + \beta_1 x_i + \beta_2 x_{i2} + ... + \beta_k x_{ik})
\]

(5)

Estimation method is used on Generalized Poisson Regression is Maximum Likelihood Estimation (MLE). Method to compute statistic test on parameter test is using Maximum Likelihood Ratio Test (MLRT):

\[
D(\hat{\beta}) = -2 \ln \left( \frac{L(\hat{\phi})}{L(\hat{\theta})} \right) = 2 \left( \ln L(\hat{\theta}) - \ln L(\hat{\phi}) \right)
\]

(6)

where

\(L(\hat{\theta})\) : Maximum Likelihood for complete model with predictor variables

\(L(\hat{\phi})\) : Maximum Likelihood for simple model without predictor variables

C. Bivariate Generalized Poisson Regression

According to [7] for example \(N_1, N_2, N_3\) are random variable and independent so Generalized Poisson distribution \(Y_i = N_i + N_j\) and \(Y_j = N_i + N_j\) Consul dan Shoukri (1985) in Vernic (1997) explained that \(X \sim \text{GDP}(\mu, \alpha)\) then the probability function of Bivariate Generalized Poisson :

\[
f(y_{1}, y_{2}) = \left\{ \begin{array}{ll}
\frac{\mu_1 \mu_2}{(\mu_1 + \mu_2 + \mu_3)^{y_{1}+y_{2}}} \exp\left[-(\mu_1 + \mu_2 + \mu_3)(y_{1}+y_{2})-\frac{\mu_2 (y_{1}+y_{2})}{\mu_3}\right] & \\
\exp\left[\frac{\mu_2 (y_{1}+y_{2})}{\mu_3}\right] & \\
\text{min}(y_{1}, y_{2}) & \end{array} \right.
\]

\[
\min(y_{1}, y_{2}) \frac{1}{(\sum_{k=0}^{y_{1}+y_{2}} (\gamma_{1}-k)!(\gamma_{2}-k)!k!\left(\gamma_{1}+(\gamma_{1}-k)\alpha_{1}\right)\gamma_{1}-k-l\left(\gamma_{2}+(\gamma_{2}-k)\alpha_{2}\right)\gamma_{2}-k-l\left(\gamma_{3}+\gamma_{2}+\gamma_{1}+k\alpha_{3}\right)^{k-l}\left(\gamma_{1}+\gamma_{2}+\gamma_{3}+k\alpha_{3}\right)^{k-l})}
\]

Model of Bivariate Generalized Poisson Regression is

\[
\mu_{ij} = \exp(x_i^T \beta_j) = \exp(\beta_{i0} + \beta_{i1} x_i + \beta_{i2} x_{i2} + ... + \beta_{ik} x_{ik})
\]

Parameter estimation of Bivariate Generalized Poisson Regression is done by using Maximum Likelihood Estimation (MLE). The method for calculating statistic test is using Maximum Likelihood Ratio Test (MLRT):

\[
D(\hat{\beta}) = -2 \ln \left( \frac{L(\hat{\phi})}{L(\hat{\theta})} \right) = 2 \left( \ln L(\hat{\theta}) - \ln L(\hat{\phi}) \right)
\]

(9)

where

\(L(\hat{\theta})\) : Maximum Likelihood for complete model with predictor variables
\( L(\hat{\theta}) \): Maximum Likelihood for simple model without predictor variables

III. METHOD

Steps to get parameter estimator Bivariate Generalized Poisson Regression model is (1) Forming the likelihood function of Bivariate Generalized Poisson Regression model; (2) Forming function In likelihood of Bivariate Generalized Poisson Regression model; (3) Transforming In likelihood function; (4) Looking first partial derivative of In likelihood function; (5) Looking for second partial derivatives of In likelihood function; (6) Getting parameter estimator with Newton Raphson iteration; (7) Hypotheses test simultaneous is using MLTR and partial test.

IV. RESULT AND DISCUSSION

Parameter estimation method Bivariate Generalized Poisson Regression is a Maximum Likelihood Estimation (MLE) with probability function:

\[
f(y_{1i}, y_{2i}) = \left\{ \begin{array}{ll}
\frac{\mu_1 \mu_2 \exp \left\{ -\left( \mu_0 + \mu_1 + \mu_2 \right) - y_{1i} \right\} \min(\mu_1 y_{1i})}{\left( \gamma_{1i} - 1 \right)!} \\
\frac{\left( \mu_2 + \left( y_{1i} - \mu_0 \right) + y_{2i} \right) \min(\mu_1 y_{1i}))}{\left( \gamma_{1i} - 1 \right)!} \left( \gamma_{1i} - 1 \right)!}
\end{array} \right.
\]

the function In likelihood of Bivariate Generalized Poisson:

\[
Q = -\ln \left( L(\mu_0, \mu_1, \mu_2) \right) = \frac{\gamma_{1i}}{\gamma_{1i}} \left( y_{1i} - \mu_0 \right) + \frac{\gamma_{1j}}{\gamma_{1j}} \left( y_{2i} - \mu_0 \right)
\]

the first derivative of \( \ln L(\mu_0, \mu_1, \mu_2) \) to \( \mu_0 \) is

\[
\frac{\partial Q}{\partial \mu_0} = \gamma_{1i} \left( y_{1i} - \mu_0 \right) + \gamma_{2i} \left( y_{2i} - \mu_0 \right)
\]

the first derivative of \( \ln L(\mu_0, \mu_1, \mu_2) \) to \( \mu_1 \) is

\[
\frac{\partial Q}{\partial \mu_1} = \gamma_{1i} \left( y_{1i} - \mu_1 \right) + \gamma_{2i} \left( y_{2i} - \mu_1 \right)
\]

the first derivative of \( \ln L(\mu_0, \mu_1, \mu_2) \) to \( \mu_2 \) is

\[
\frac{\partial Q}{\partial \mu_2} = \gamma_{1i} \left( y_{1i} - \mu_2 \right) + \gamma_{2i} \left( y_{2i} - \mu_2 \right)
\]

the first derivative of \( \ln L(\mu_0, \mu_1, \mu_2) \) to \( \alpha_1 \) is

\[
\frac{\partial Q}{\partial \alpha_1} = \gamma_{1i} \left( y_{1i} - \mu_1 \right) + \gamma_{2i} \left( y_{2i} - \mu_1 \right)
\]

the first derivative of \( \ln L(\mu_0, \mu_1, \mu_2) \) to \( \alpha_2 \) is

\[
\frac{\partial Q}{\partial \alpha_2} = \gamma_{1i} \left( y_{1i} - \mu_2 \right) + \gamma_{2i} \left( y_{2i} - \mu_2 \right)
\]
the first derivative of \( \ln L(\theta) \) to \( \alpha_2 \) is

\[
\frac{\partial Q}{\partial \alpha_2} = -\sum_{i=1}^{n} y_{2i} + \sum_{k=1}^{\min(y_{2i})} \left( y_{2i} - k \right) \left( y_{2i} - k - 1 \right) \left( e^{\beta_{\sum_{j=1}^{k} y_{2j}} - \beta_0} \right) + \left( y_{2i} - k \right) \alpha_2 \]

(14)

the first derivative does not close form so that to solve the equation is using Newton-Raphson iteration with equation (15) as follow

\[
\hat{\theta}_{(m+1)} - \hat{\theta}_{(m)} - H^{-1} \left( \frac{\partial \ln L(\theta)}{\partial \theta} \right) \]

(15)

where

\[
\hat{\theta}_{(m)} = \left( \hat{\mu}_0, \hat{\beta}_1^T, \hat{\beta}_2^T, \alpha_1, \alpha_2 \right)^T
\]

(16)

\[
H(\theta_{(m)}) = \begin{bmatrix}
\frac{\partial^2 \ln L(\theta)}{\partial \mu_0^2} & \frac{\partial^2 \ln L(\theta)}{\partial \mu_0 \beta_1} & \frac{\partial^2 \ln L(\theta)}{\partial \mu_0 \beta_2} & \frac{\partial^2 \ln L(\theta)}{\partial \mu_0 \alpha_1} & \frac{\partial^2 \ln L(\theta)}{\partial \mu_0 \alpha_2}\\
\frac{\partial^2 \ln L(\theta)}{\partial \beta_1 \mu_0} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_1 \beta_1} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_1 \beta_2} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_1 \alpha_1} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_1 \alpha_2}\\
\frac{\partial^2 \ln L(\theta)}{\partial \beta_2 \mu_0} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_2 \beta_1} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_2 \beta_2} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_2 \alpha_1} & \frac{\partial^2 \ln L(\theta)}{\partial \beta_2 \alpha_2}\\
\frac{\partial^2 \ln L(\theta)}{\partial \alpha_1 \mu_0} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_1 \beta_1} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_1 \beta_2} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_1 \alpha_1} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_1 \alpha_2}\\
\frac{\partial^2 \ln L(\theta)}{\partial \alpha_2 \mu_0} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_2 \beta_1} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_2 \beta_2} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_2 \alpha_1} & \frac{\partial^2 \ln L(\theta)}{\partial \alpha_2 \alpha_2}
\end{bmatrix}
\]

(17)

Hessian matrix is matrix that contain second derivative of function \( \ln L(\theta) \) to the parameter \( \left( \mu_0, \beta_1^T, \beta_2^T, \alpha_1, \alpha_2 \right) \)

Parameter estimation steps with Newton-Raphson iteration is

1. Determining the initial value of parameter
2. Forming vector \( g(\theta) \) by substituting equation (10), (11), (12), (13) and (14) into the equation (15)
3. Hessian matrix forming by substitute the resulting equation of second derivative into (17).
4. Inserting values into \( \hat{\theta}_{(0)} \) vector elements and matrix \( H \)
5. Starting from \( m = 0 \) iterating at the equation (15). Value \( \hat{\theta}_{(m)} \) is a collection of parameter estimator which convergent the current iteration \( m \).
6. If have not gotten parameter estimation are convergent, then proceeded to step 5 to \( m = m + 1 \) iteration. Iteration will stop if the value of \( \left\| \hat{\theta}_{(m+1)} - \hat{\theta}_{(m)} \right\| \leq \varepsilon \).

Hypothesis test is done in two parts that is

i. Hypothesis testing simultaneous is using MLTR with hypotheses

Parameter \( \beta \)

\[ H_0 : \beta_{j1} = \beta_{j2} = \cdots = \beta_{jk} = \beta_{j}; j = 1, 2 \]

\[ H_1 : \text{at least one } \beta_{jk} \neq 0; j = 1, 2 \text{ with } l = 1, 2, \ldots, k \]

Parameter \( \alpha \)

\[ H_0 : \alpha_1 = \alpha_2 = 0; \]

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$H_1$: at least one $\alpha_j \neq 0; j = 1, 2$

Statistical test:

$$D(\hat{\beta}) = -2\ln\left(\frac{L(\hat{\omega})}{L(\hat{\alpha})}\right) = 2\left(\ln L(\hat{\alpha}) - \ln L(\hat{\omega})\right)$$

$L(\hat{\alpha})$: Maximum Likelihood for complete model with predictor variables and $L(\hat{\omega})$: Maximum Likelihood for simple model without predictor variables. The rejection region $H_0$ if

$$D(\hat{\beta}) > \chi^2(\alpha, v)$$

ii. Hypothesis testing partial is using $Z$-test with hypothesis

Parameter $\beta$

$$H_0: \beta_{jl} = 0$$

$$H_1: \beta_{jl} \neq 0 \text{ with } j = 1, 2 \text{ and } l = 1, 2, ..., k$$

Statistical test:

$$Z_{cal} = \frac{\hat{\beta}_{jl}}{SE(\hat{\beta}_{jl})}$$

The rejection region $H_0$ if $|Z_{cal}| > Z(\alpha, v)$

Parameter $\alpha$

$$H_0: \alpha_j = 0$$

$$H_1: \alpha_j \neq 0 \text{ with } j = 1, 2$$

Statistical test:

$$Z_{cal} = \frac{\hat{\alpha}_j}{SE(\hat{\alpha}_j)}$$

The rejection region $H_0$ if $|Z_{cal}| > Z(\alpha, v)$

RESULT

Parameter estimation model on Bivariate Generalized Poisson Regression is using Maximum Likelihood (ML). The results obtained from parameter estimation does not close form that needs to be done by Newton Raphson iteration. On hypothesis test is using Maximum Likelihood Ratio Test (MLRT) by comparing the value of the likelihood under $H_0$ and likelihood under population. Hypothesis test is done in two parts simultaneous and partial.

REFERENCES
