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Global Stability of SACR Epidemic Model for Hepatitis C on Injecting Drug Users

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***Abstract*—** This study aims to analyze SACR model of epidemics of hepatitis C on injecting drug users. This model is represented by a system of nonlinear differential equations based on model that written by Kretzschmar M, Wiessing L (2004). In this paper will be discussed detail about the global stability of the disease free equilibrium point. The analysis results obtained to equilibrium points that is disease free and endemic equilibrium point. The Disease-free equilibrium point is local asymptotically stable if the basic reproduction number is less than one, and it is global asymptotically stable if the basic reproduction number is less than or equal to one. It means that for a long period of time, the infected population will decrease or even disappear so that the virus no longer exists in the population. Meanwhile, for the basic reproduction number is more than one, the disease-free equilibrium point is unstable and the endemic equilibrium point is local asymptotically stable. This shows that for a certain period, the Hepatitis C virus will be persist. Furthermore, based on the simulation of this model, it is found that the higher the average frequency of the use of needles together in injecting drug users, the more of the proportion of acute infection, chronic carriers and recovered. While the proportion of susceptible decreased.

Keywords: *Hepatitis C, injecting drug users, the SACR model, equilibrium points, global stability.*

I. INTRODUCTION

Epidemic models commonly used in analyzing the spread of disease is a SIR model. Based on its characteristics, this model classifying the population into three subpopulations. They are susceptible (groups of individuals free infected disease), infected (groups of individuals infected disease), and recovered (group of individuals who have been cleared of the disease). In certain diseases, some infected individuals can develop into chronic, so they need for a developing model that is able to accommodate the characteristics of the disease, such as adding a group such as carrier subpopulation.

On carrier states, susceptible individuals can be infected by contact with acute infection individual and carrier individual. Acute infection individual in a period will be recovered totally by itself, or it can be developed into a carrier virus. [1]

One of the disease which can be analyzed with the model SACR is the spread of Hepatitis C. Generally, this model can be applied to the spread of Hepatitis C because of an individual with acute infection Hepatitis C virus can be develop into chronic Hepatitis or will recovery by itself (though in small percentage). In this study the spread of Hepatitis C is focused on injecting drug user community. This is because the percentage of the case is high.

Based on the World Health Organization (WHO) data in 2009 [2], it is estimated that seven million people in Indonesia suffer Hepatitis C virus and thousands of new infections occur annually, but 90% of people are not aware of that condition. The disease can be transmitted through blood contact between individuals, and the highest risk occurs when using injecting together among drug users community. In Indonesia, the prevalence of HCV among injecting drug users reached 77.3%. [3]

Some references that support this research are project research about epidemic model with carrier population by Maia Martcheva and Carlos Castillo-Chavez (2003) [4] entitled *Diseases with chronic stage in a population with varying size*. Research about *Modelling the transmission of hepatitis C in injecting drug users* has been done by M.Kretzschmar and L. Wiessing (2004) [5]. And Dontwi, et al (2010) [6] *Mathematical modeling of Hepatitis C Virus transmission among injecting drug users and the*

impact of vaccination. In this research will be discussed about stability analyzis of the disease free equilibrium point using a model that is developed by M. Kretzschmar and L. Wiessing (2004) [5].

II. MODEL FORMULATION

Individuals will be entered into the population because of their birth or their recruitment and leave the population through death or emigration. The total population is all individuals are susceptible, infected by hepatitis C virus or who have been recovered from hepatitis C. In this case, in susceptible individuals infected are injecting drug users. The rate of recruitment of new injecting drug users determine raising of susceptible individuals (injecting drug users) in the population. Individuals are easy to be infected with hepatitis C acute when contact with an infected individual acute or chronic. The infection rate is influenced by how often the contacts or share needles among injecting drug users. Individuals are more likely to use inject together will be more at risk to be infected with hepatitis C. Patients with hepatitis C are in the acute phase of approximately 6-10 weeks. Most people with acute infection will develop into chronic, whereas the part of them will recoverl by themself. Patients with acute hepatitis C will develop chronic liver within 15-20 years, or can also be liver cancer after 20-30 years, or it will die. Almost all of mortality from patients with hepatitis C-related complications of cirrhosis and liver cancer, so mortality due to hepatitis C virus infection is very small.

In this study, the human population at time *t* divided into four subpopulations. They are susceptible, acute infection, chronic carrier (chronically infected), and recovered (free of hepatitis C).

To simplify the model is given the following assumptions:

1. The population is constant and closed,
2. The population is homogeneous, it means that everyone has the same risk for infected the virus and the frequency of use inject together (non-sterile) is constant,
3. Individuals who haven't been infected disease include susceptible class,
4. The mortality due to hepatitis C virus infection is ignored and it just happened natural mortality in each subpopulation.
5. Individuals who have recovered from hepatitis C, will not be infected again and become have immune to the virus Hepatitis C.

The variables and parameters which used in this study:

- $S(t)$: The number of *susceptible* individuals
- $A(t)$: The number of *acute infection* individuals
- $C(t)$: The number of *chronic carrier* individuals
- $R(t)$: The number of *recovered* individuals
- $N(t)$: Total populations
- B: The rate of recruitment
- λ: The rate of infection
- κ: Average frequency of use inject (needles) together
- b_a: Probability transmission as a result of contact between susceptible individuals to acute infection individuals
- b_c: Probability transmission as a result of contact between susceptible individuals with chronic individual carrier
- μ: Natural mortality rate
- σ_1: Individual transfer rate of acute infection becomes chronic individual carrier
- σ_2: Individual transfer rate of chronic carrier into individual Recovered
- ρ: The proportion of acute infection who become chronic carriers
- $(1-\rho)$: The proportion of acute infection is completely cured

with $S(t), A(t), C(t), R(t) \geq 0$ *and* $B, \lambda, \kappa, b_a, b_c, \mu, \sigma_1, \sigma_2, \rho, (1-\rho) > 0$.

Based on the characteristics and the assumptions, the spread of hepatitis C can be described in the following flow chart:

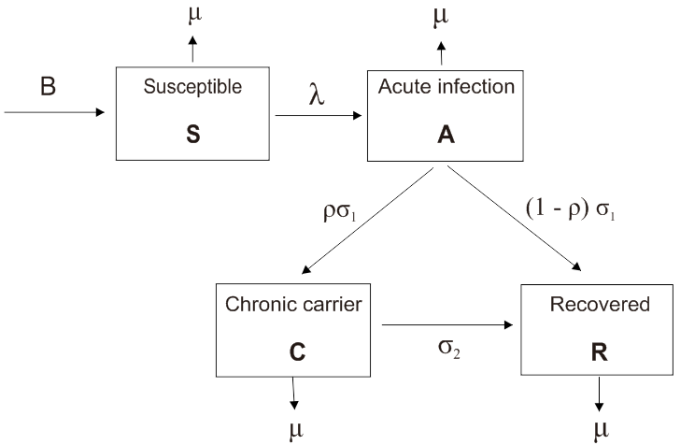

Figure 1. Flow diagram of the spread of hepatitis C in injecting drug users

Of the flow chart above, obtained by differential equations system is a model the spread of hepatitis C in injecting drug users. [5]

$$\begin{aligned}\frac{dS(t)}{dt} &= B - \lambda S(t) - \mu S(t)\\ \frac{dA(t)}{dt} &= \lambda S(t) - \sigma_1 A(t) - \mu A(t)\\ \frac{dC(t)}{dt} &= \rho\sigma_1 A(t) - \sigma_2 C(t) - \mu C(t)\\ \frac{dR(t)}{dt} &= (1-\rho)\sigma_1 A(t) + \sigma_2 C(t) - \mu R(t)\end{aligned} \quad (1)$$

with $\lambda(t) = \kappa\left(b_a \frac{A(t)}{N(t)} + b_c \frac{C(t)}{N(t)}\right)$ and $N = \frac{B}{\mu}$

Furthermore, to simplify the system, set up a proportion which compares the number of individuals in a subpopulation with the number of total population, we get

$$\begin{aligned}\frac{ds}{dt} &= \mu - \lambda(t)s(t) - \mu s(t)\\ \frac{da}{dt} &= \lambda(t)s(t) - \sigma_1 a(t) - \mu a(t)\\ \frac{dc}{dt} &= \rho\sigma_1 a(t) - \sigma_2 c(t) - \mu c(t)\\ \frac{dr}{dt} &= (1-\rho)\sigma_1 a(t) + \sigma_2 c(t) - \mu r(t)\end{aligned} \quad (2)$$

with $\lambda(t) = \kappa\big(b_a a(t) + b_c c(t)\big)$.

In this model use a parameter stating the expected value of a new case caused by infected individuals in a population of susceptible individuals. These parameters are the basic reproduction number (R_0). Based on the model (2) defined parameter

$$R_0 = \kappa\left(\frac{b_a}{(\sigma_1+\mu)} + b_c \frac{\rho\sigma_1}{(\sigma_2+\mu)(\sigma_1+\mu)}\right).$$

III. EQUILIBRIUM POINT MODEL ANALYSIS

These are equilibrium points of the system (2) are described in **theorem 1**.

Theorem 1.

(i) If $a = 0$ *then system (2) has an equilibrium point that is disease-free,* $E_0 = (s, a, c, r) = (1,0,0,0)$.

(ii) If $a \neq 0$ then system(2) has the endemic equilibrium point $E_1 = (\hat{s}, \hat{a}, \hat{c}, \hat{r})$.

Proof:

The *(s,a,c,r)* point is the equilibrium point system (2) if $\frac{ds}{dt} = \frac{da}{dt} = \frac{dc}{dt} = \frac{dr}{dt} = 0$ [7]. System (2) can be written as

$$\begin{aligned} &\mu - \lambda s - \mu s = 0 && (2a)\\ &\lambda s - \sigma_1 a - \mu a = 0 && (2b)\\ &\rho\sigma_1 a - \sigma_2 c - \mu c = 0 && (2c)\\ &(1-\rho)\sigma_1 a + \sigma_2 c - \mu r = 0 && (2d) \end{aligned}$$

with $\lambda(t) = \kappa(b_a a + b_c c)$.

(i) If $a = 0$, then from equation (2c) is obtained $c = 0$. If $a = 0$ dan $c = 0$ are substituted into the equation (2a) and (2d) obtained $s = 1$ dan $r = 0$. This proves that the disease-free equilibrium point is $E_0 = (1,0,0,0)$.

(ii) If $a \neq 0$ (which symbolized by $\hat{a}$), then the equation (2c) becomes $\rho\sigma_1\hat{a} - \sigma_2\hat{c} - \mu\hat{c} = 0$ or $\hat{c} = \frac{\rho\sigma_1}{(\sigma_2+\mu)}\hat{a}$. Furthermore, if the equation $\hat{c}$ is substituted into the equation (2b) obtained $\hat{s} = \frac{(\sigma_1+\mu)}{\kappa\left(b_a + b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)}$. Consequently, from equation (2a) is obtained $\hat{a} = \frac{\mu\left(\kappa\left(b_a+b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)-(\sigma_1+\mu)\right)}{\kappa\left(b_a+b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)(\sigma_1+\mu)}$. Then from equation (2d) is obtained $\hat{r} = \left(\frac{(1-\rho)\sigma_2+\mu}{\mu(\sigma_2+\mu)}\right)\sigma_1\hat{a}$. So, this proves that the endemic equilibrium point is $E_1 = (\hat{s}, \hat{a}, \hat{c}, \hat{r})$with

$$\hat{s} = \frac{(\sigma_1+\mu)}{\kappa\left(b_a + b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)}; \qquad \hat{a} = \frac{\mu\left(\kappa\left(b_a + b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right) - (\sigma_1+\mu)\right)}{\kappa\left(b_a + b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)(\sigma_1+\mu)}$$

$$\hat{c} = \frac{\rho\sigma_1}{(\sigma_2+\mu)}\hat{a}; \qquad \hat{r} = \frac{(\sigma_2+(1-\rho)\mu)\sigma_1}{\mu(\sigma_2+\mu)}\hat{a}.$$

IV. STABILITY ANALYSIS OF EQUILIBRIUM POINTS

In this section, will be described analysis of global stability of the equilibrium point of system (2). The stability of the equilibrium point system (2) is presented in Theorem 2, Theorem 3 and Theorem 4 as follow

Theorem 2.

(i) If $R_0 < 1$ then the disease-free equilibrium point $E_0 = (s,a,c,r) = (1,0,0,0)$ is local asymptotically stable and

(ii) If $R_0 > 1$ then the disease-free equilibrium point $E_0 = (s,a,c,r) = (1,0,0,0)$ is unstable.

Proof

The Jacobian matrix of the system (2) around the equilibrium point is$E_0 = (1,0,0,0)$ is

$$J(E_0) = \begin{bmatrix} -\kappa(b_a a + b_c c) - \mu & -\kappa b_a s & -\kappa b_c s & 0 \\ \kappa(b_a a + b_c c) & \kappa b_a s - \sigma_1 - \mu & \kappa b_c s & 0 \\ 0 & \rho\sigma_1 & -\sigma_2 - \mu & 0 \\ 0 & (1-\rho)\sigma_1 & \sigma_2 & -\mu \end{bmatrix} \qquad (3)$$

The characteristic equation of the (3) can be searched by specifying $det(J(E_0) - \gamma I) = 0$, with γ is eigenvalue and *I* is the identity matrix. Thus obtained:

$$(\gamma+\mu)^2\left(\gamma^2 + \left(-\kappa b_a + (\sigma_2+\mu) + (\sigma_1+\mu)\right)\gamma - \rho\sigma_1\kappa b_c - \kappa b_a(\sigma_2+\mu) + (\sigma_2+\mu)(\sigma_1+\mu)\right) = 0. \qquad (4)$$

The equation (4) can be written as

$$(\gamma+\mu)^2(\gamma^2 + q_1\gamma + q_2) = 0$$

with

$$q_1 = -\kappa b_a + (\sigma_2+\mu) + (\sigma_1+\mu)$$
$$q_2 = -\rho\sigma_1\kappa b_c - \kappa b_a(\sigma_2+\mu) + (\sigma_2+\mu)(\sigma_1+\mu).$$

Based on the equation (4), it is obtained eigenvalues$\gamma_1 = \gamma_2 = -\mu$. For other eigenvalues, Routh-Hurwitz criteria used to expand the type of stability of the characteristic equation [8]

$$\gamma^2 + q_1\gamma + q_2 = 0. \quad (5)$$

The equation$q_1 = -\kappa b_a + (\sigma_2 + \mu) + (\sigma_1 + \mu)$can be expressed as

$$q_1 = \frac{b_a(\sigma_1 + \mu)(\sigma_2 + \mu)(1-R_0) + (\sigma_2 + \mu)(b_a(\sigma_2 + \mu) + b_c\rho\sigma_1) + b_c\rho\sigma_1(\sigma_1 + \mu)}{b_a(\sigma_2 + \mu) + b_c\rho\sigma_1}$$

Furthermore, the equation$q_2 = -\rho\sigma_1\kappa b_c - \kappa b_a(\sigma_2 + \mu) + (\sigma_2 + \mu)(\sigma_1 + \mu)$can be expressed as$q_2 = (R_0 + 1)(\sigma_1 + \mu)(\sigma_2 + \mu)$.

It is known that$R_0 < 1$, then$q_1 > 0$ and$q_2 > 0$. Based on Routh Hurwitz criteria, zero maker from the equation (5) will be negative if$\mathrm{q}_1 > 0$ and$\mathrm{q}_2 > 0$. This shows that all eigenvalues of equation (4) isnegative, so that the disease-free equilibrium point $\mathrm{E}_0 = (1,0,0,0)$is islocal asymptotically stable.

Meanwhile, if known$R_0 > 1$, then it is obtained$q_2 < 0$. The roots of the equation (5) will be different from that mark$\gamma_1 < 0$ and$\gamma_2 > 0$or the opposite. Thus, it can be said that if$R_0 > 1$, then the equation (4) has a positive eigenvalues. Therefore, the disease-free equilibrium point $E_0 = (s, a, c, r) = (1,0,0,0)$is unstable.

Theorem 3.

If $R_0 \leq 1$ then disease-free equilibrium point $E_0 = (s, a, c, r) = (1,0,0,0)$ is global asymptotically stable.

Proof.

To express the global stability of the disease-free equilibrium point, it will be shown $s(t) \to 1$, $a(c) \to 0$, $c(t) \to 0$, $r(t) \to 0$ at time $t \to \infty$. Firstly, determine solutions of differential equations in system (2).

The second equation solution of the system (2) is

$$a(t) = e^{-(\sigma_1+\mu)t}\int e^{(\sigma_1+\mu)t}\lambda(t)s(t)\,dt + a_0e^{-(\sigma_1+\mu)t}$$

Because$0 \leq s(t) \leq 1$, then

$$a(t) \leq e^{-(\sigma_1+\mu)t}\int e^{(\sigma_1+\mu)t}\lambda(t)\,dt + a_0e^{-(\sigma_1+\mu)t}$$

$$= \int_0^t e^{-(\sigma_1+\mu)(t-\tau)}\lambda(\tau)\,d\tau + a_0e^{-(\sigma_1+\mu)t}$$

Furthermore, by taking lim sup for $t \to \infty$ [9], it is obtained

$$\limsup_{t\to\infty} a(t) \leq \limsup_{t\to\infty}\int_0^t e^{-(\sigma_1+\mu)\tau}\lambda(t-\tau)\,d\tau$$

$$\leq \limsup_{t\to\infty}\lambda(t)\int_0^\infty e^{-(\sigma_1+\mu)\tau}\,d\tau$$

$$= \frac{1}{(\sigma_1+\mu)}\limsup_{t\to\infty}\lambda \quad (6)$$

Then determined solutions to a third equation of the system (2) that

$$c(t) = \rho\sigma_1e^{-(\sigma_2+\mu)t}\int e^{(\sigma_2+\mu)t}a\,dt + c_0e^{-(\sigma_2+\mu)t}$$

Furthermore, by taking lim sup for$t \to \infty$, it is obtained

$$\limsup_{t\to\infty} c(t) \leq \rho\sigma_1\limsup_{t\to\infty}\int_0^t e^{-(\sigma_2+\mu)\theta}a(t-\theta)\,d\theta$$

$$\leq \rho\sigma_1\limsup_{t\to\infty} a\,(t)\int_0^\infty e^{-(\sigma_2+\mu)\theta}\,d\theta$$

$$= \frac{\rho\sigma_1}{(\sigma_2+\mu)}\limsup_{t\to\infty} a\,(t). \quad (7)$$

Then inequality (6) is substituted into the inequality (7) is obtained

$$\limsup_{t\to\infty} c(t) \leq \frac{\rho\sigma_1}{(\sigma_1+\mu)(\sigma_2+\mu)}\limsup_{t\to\infty}\lambda(t) \quad (8)$$

Then we wil discuss the rate of hepatitis C virus infection. It will be demonstrated that $\lim_{t\to\infty}\sup\lambda(t) = 0$, by taking lim sup for$t \to \infty$, then

$$\limsup_{t\to\infty} \lambda\,(t) = \limsup_{t\to\infty} \left(\kappa\big(b_a a(t) + b_c c(t)\big)\right).$$

Substitution inequality (6) and (8) into the above equation is obtained

$$\begin{aligned}\limsup_{t\to\infty} \lambda &\le \kappa b_a \frac{1}{(\sigma_1+\mu)} \limsup_{t\to\infty} \lambda(t) + \kappa b_c \frac{\rho\sigma_1}{(\sigma_1+\mu)(\sigma_2+\mu)} \limsup_{t\to\infty} \lambda(t)\\ &= \left(\frac{\kappa b_a}{(\sigma_1+\mu)} + \frac{\kappa b_c \rho\sigma_1}{(\sigma_1+\mu)(\sigma_2+\mu)}\right) \limsup_{t\to\infty} \lambda\\ &= R_0 \limsup_{t\to\infty} \lambda.\end{aligned}$$

then

$$\limsup_{t\to\infty} \lambda \le R_0 \limsup_{t\to\infty} \lambda$$

$$\limsup_{t\to\infty} \lambda - R_0 \limsup_{t\to\infty} \lambda \le 0$$

$$\limsup_{t\to\infty} \lambda\,(1 - R_0) \le 0.$$

as it is known $R_0 \le 1$, then $\lim_{t\to\infty} \sup \lambda = 0$.

Furthermore, with $\lim_{t\to\infty} \sup \lambda = 0$, Then based inequalities (5) and (8) were obtained $\lim_{t\to\infty} \sup a(t) = 0$ dan $\lim_{t\to\infty} \sup c(t) = 0$

And then determined the solution to fourth equations of the system (2) that is

$$r(t) = (1-\rho)\sigma_1 \int_0^t e^{-\mu\omega} a(t-\omega)\, d\omega + \sigma_2 \int_0^t e^{-\mu\omega} c(t-\omega)\, d\omega + r_0 e^{-\mu t}$$

Furthermore, by taking lim sup for $t \to \infty$, it is obtained

$$\begin{aligned}\limsup_{t\to\infty}\ r(t) &\le (1-\rho)\sigma_1 \limsup_{t\to\infty} \int_0^t e^{-\mu\omega} a(t-\omega)\, d\omega + \sigma_2 \limsup_{t\to\infty} \int_0^t e^{-\mu\omega} c(t-\omega)\, d\omega\\ &\le \frac{(1-\rho)\sigma_1}{\mu} \limsup_{t\to\infty}\ a(t) + \frac{\sigma_2}{\mu} \limsup_{t\to\infty} c(t) \qquad (9)\end{aligned}$$

Because $\lim_{t\to\infty} \sup\ a(t) = \lim_{t\to\infty} \sup\ c(t) = 0$, then from inequality (9) is obtained $\lim_{t\to\infty} \sup\ r(t) = 0$. Thus obtained that $\lim_{t\to\infty} a(t) = 0$, $\lim_{t\to\infty} c(t) = 0$ and $\lim_{t\to\infty} r(t) = 0$.

It is known that $s(t) + a(t) + c(t) + r(t) = 1$, by taking the limit value $t \to \infty$, then obtained

$$\lim_{t\to\infty}\big(s(t) + a(t) + c(t) + r(t)\big) = \lim_{t\to\infty} 1$$

$$\lim_{t\to\infty} s(t) + \lim_{t\to\infty} a(t) + \lim_{t\to\infty} c(t) + \lim_{t\to\infty} r(t) = 1$$

Because $\lim_{t\to\infty} a\,(t) = 0, \lim_{t\to\infty} c(t) = 0$, and $\lim_{t\to\infty} r(t) = 0$ then $\lim_{t\to\infty} s(t) = 1$. Thus it is proved for $R_0 \le 1$ then the disease-free equilibrium point $E_0 = (1,0,0,0)$ is global asymptotically stable.

Theorem 4

If $R_0 > 1$ then the endemic equilibrium point $E_1 = (\hat{s}, \hat{a}, \hat{c}, \hat{r})$ is local asymtotically stable.

Proof:

Jacobian matrix of the system (2) around the equilibrium point $E_1 = (\hat{s}, \hat{a}, \hat{c}, \hat{r})$ is

$$J(E_1) = \begin{bmatrix} -\kappa(b_a\hat{a} + b_c\hat{c}) - \mu & -\kappa b_a \hat{s} & -\kappa b_c \hat{s} & 0\\ \kappa(b_a\hat{a} + b_c\hat{c}) & \kappa b_a \hat{s} - \sigma_1 - \mu & \kappa b_c \hat{s} & 0\\ 0 & \rho\sigma_1 & -\sigma_2 - \mu & 0\\ 0 & (1-\rho)\sigma_1 & \sigma_2 & -\mu \end{bmatrix} \qquad (10)$$

Suppose

$$x = \kappa(b_a\hat{a} + b_c\hat{c}) = \kappa\left(b_a + \frac{b_c\rho\sigma_1}{(\sigma_2+\mu)}\right)\hat{a}\,; \quad y = \kappa b_a \hat{s} = \frac{\kappa b_a(\sigma_1+\mu)}{\kappa\left(b_a + b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)}$$

$$z = \kappa b_c \hat{s} = \frac{\kappa b_c(\sigma_1+\mu)}{\kappa\left(b_a + b_c\frac{\rho\sigma_1}{(\sigma_2+\mu)}\right)}; \qquad m = (\sigma_1+\mu); \qquad n = (\sigma_2+\mu)$$

then the equation (10) can be expressed as

$$J(E_1) = \begin{bmatrix} -x-\mu & -y & -z & 0\\ x & y-m & z & 0\\ 0 & \rho\sigma_1 & -n & 0\\ 0 & (1-\rho)\sigma_1 & \sigma_2 & -\mu \end{bmatrix}$$

The characteristic equation of the equation (10) can be obtained by solving $det(J(E_2) - \gamma I) = 0$, with γ is an eigen value and I is the identity matrix. Thus, it is obtained:

$$(\mu + \gamma)\big(\gamma^3 + (n + x + \mu - y + m)\gamma^2 + (xn + \mu n - yn + mn + xm - \mu x + \mu m - z\rho\sigma_1)\gamma + (xnm - \mu yn + \mu mn - \rho\sigma_1 z\mu)\big) = 0$$

or can be expressed as

$$(\mu + \gamma)(\gamma^3 + k_1\gamma^2 + k_2 y + k_3) = 0 \qquad (11)$$

with

$k_1 = n + x + \mu - y + m$
$k_2 = xn + \mu n - yn + mn + xm - \mu y + \mu m - z\rho\sigma_1$
$k_3 = xnm - \mu yn + \mu mn - \rho\sigma_1 z\mu$

Based on the equation (11), it is obtained eigen value $\gamma_1 = -\mu$. For the others eigenvalues, will be used Routh-Hurwitz criteria to look at the roots of the characteristic equation $(\gamma^3 + k_1\gamma^2 + k_2 y + k_3)$. By substituting *x*, *y*, *z*, *m*, *n* into the equation k_1, k_2, and k_3, it is obtained:

$$k_1 = (\sigma_2 + \mu) + \mu R_0 + \frac{\kappa b_c \rho\sigma_1}{R_0(\sigma_2 + \mu)};$$

$$k_2 = \mu R_0(\sigma_2 + \mu) + \mu(R_0 - 1)(\sigma_1 + \mu) + \frac{\mu\kappa b_c \rho\sigma_1}{R_0(\sigma_2 + \mu)};$$

$$k_3 = \mu(R_0 - 1)(\sigma_1 + \mu)(\sigma_2 + \mu)$$

Based on the criteria of Routh-Hurwitz, zero maker of the equation$\gamma^3 + k_1\gamma^2 + k_2 y + k_3 = 0$ will be negatife if $k_1, k_2, k_3 > 0$ and $\Delta_2 = k_1k_2 - k_0k_3 > 0$. Because it was known $R_0 > 1$, then $k_1 > 0$, $k_2 > 0$, and $k_3 > 0$. Then,

$$\Delta_2 = k_1k_2 - k_0k_3$$
$$= (\sigma_2 + \mu)\left(\mu R_0(\sigma_2 + \mu) + \frac{\mu\kappa b_c \rho\sigma_1}{R_0(\sigma_2 + \mu)}\right) + \left(\mu R_0 + \frac{\kappa b_c \rho\sigma_1}{R_0(\sigma_2 + \mu)}\right)\left(\mu R_0(\sigma_2 + \mu) + \mu(R_0 - 1)(\sigma_1 + \mu) + \frac{\mu\kappa b_c \rho\sigma_1}{R_0(\sigma_2 + \mu)}\right)$$

Because of $R_0 > 1$, then$\Delta_2 = k_1k_2 - k_0k_3 > 0$.

Thus, it is obtained all eigenvalues of equation (11) is negative, so it is proved that if $R_0 > 1$, then the endemic equilibrium point $E_1 = (\hat{s}, \hat{a}, \hat{c}, \hat{r})$is local asymptotically stable.

V. MODEL SIMULATION

In this section will discusses the numerical simulation of the model to provide a simulation of the spread of hepatitis C model in injecting drug users by using MAPLE 15. Given the values of the following parameters $b_a = 0{,}3, b_c = 0{,}03$ [5], $\mu = 0{,}056$, $\frac{B}{N} = 0{,}056$, $\sigma_1 = 5, \sigma_2 = 0{,}05, \rho = 0{,}8$ and some specific initial value.

To simulate the conditions $R_0 < 1$, the value is taken $\kappa = 3$ and given initial values $s(0) = 0{,}002$, $a(0) = 0{,}008$, $c(0) = 0{,}45$, $r(0) = 0{,}54$. It is obtained,

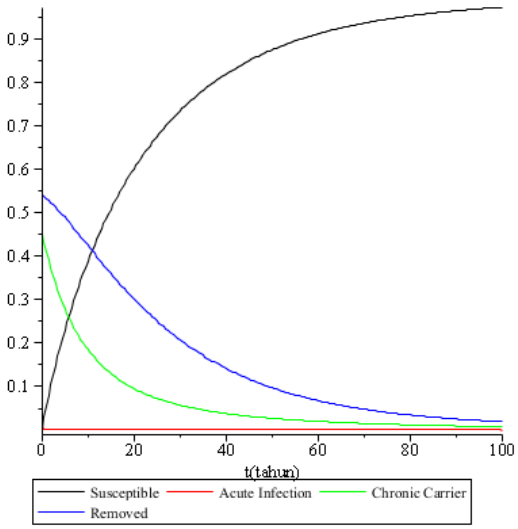

Figure 2. Simulation of the system (3.7) for$R_0 = 0{,}8497283259 < 1$

Furthermore, given also some initial values for each proportion of individuals susceptible, acute infection, chronic carrier as follows (0, 0.06, 0.015, 0.2), (0, 0.14, 0.25, 0.3), and (0, 0.49, 0.2, 0.1) to obtain a phase portrait of system solutions in the field of s, a, c. Based on these parameter values and initial values given phase portraits obtained in the field s, a, c that is shown in Figure 2.

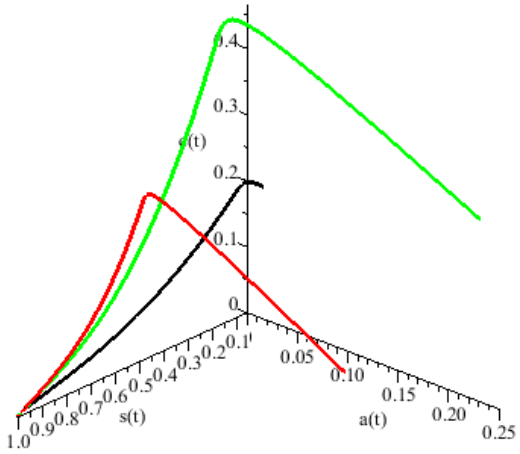

Figure3. Phase portrait in the fields,a,c for$R_0 = 0{,}8497283259 < 1$

Based on Figure 2 and Figure 3 shows that the proportion of susceptible individuals is increasing to the point 1 over time. As for the proportion of individuals acute infection, chronic (chronic carrier), and free (recovered) from Hepatitis C virus decreases towards zero. The decline in the proportion of individuals in acute infection and chronic carrier to the point of zero indicates that the proportion of individuals in acute infection and chronic carrier is die out (no infection). This simulation shows that some different initial values and when $R_0 < 1$ solution of the system (2) moves toward the equilibrium point E_0. So it can be said that when $R_0 < 1$, hepatitis C will disappear from the population.

Then, for $R_0 > 1$, given the initial value $s(0) = 0{,}8995$, $a(0) = 0{,}0005$, $c(0) = 0{,}05$, $r(0) = 0{,}05$ to obtain a phase portrait projection solutions proportion of susceptible individuals, acute infection, chronic carrier, recovered against time t as in Figure 4, Figure 5 and Figure 6.

The numerical value for E_1 when $\kappa = 6$ adalah $\hat{s} = 0{,}588423364$, $\hat{a} = 0{,}004558601980$, $\hat{c} = 0{,}1720227162$, $\hat{r} = 0{,}2349953177$. For $\kappa = 8$ obtain $\hat{s} = 0{,}4413175231$, $\hat{a} = 0{,}006187938827$, $\hat{c} = 0{,}2335071255$, $\hat{r} = 0{,}3189874126$. Meanwhile, for $\kappa = 10$ obtained $R_0 = 2{,}832427753$, $\hat{s} = 0{,}3530540184$, $\hat{a} = 0{,}007165540935$, $\hat{c} = 0{,}2703977711$, $\hat{r} = 0{,}3693826695$.

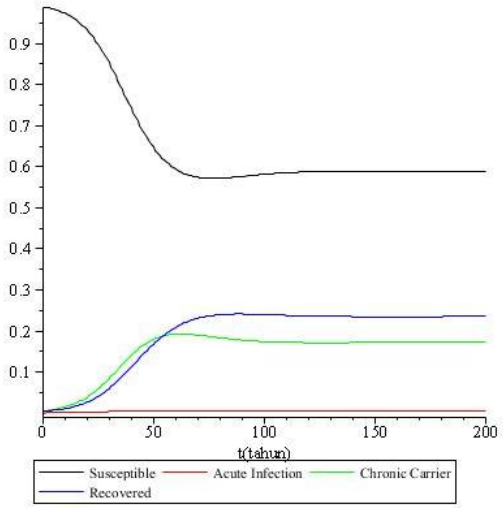

Figure 4.Simulation System (2) for$R_0 = 1{,}699456652$ with$\kappa = 6$

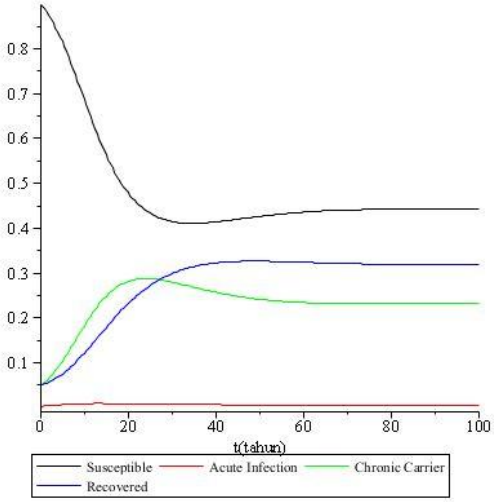

Figure 5.Simulation System (2) for$R_0 = 2{,}265942202$ with $\kappa = 8$

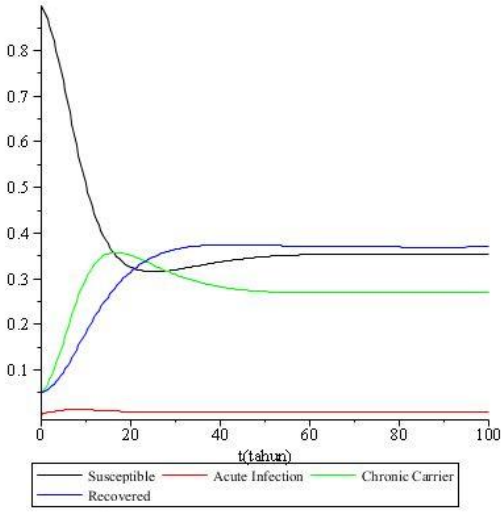

Figure 6.Simulation System (2) for$R_0 = 2{,}832427753$ with $\kappa = 10$

Furthermore, given some initial values for for each proportion of individuals susceptible, acute infection and chronic carrier are (0, 0.06, 0.015, 0.2), (0, 0.14, 0.25, 0.3), and (0, 0.49, 0.2, 0.1). To show the behavior solution of the system (2) around the endemic equilibrium point with parameter values $\kappa = 8$, can be seen in Figure 7.

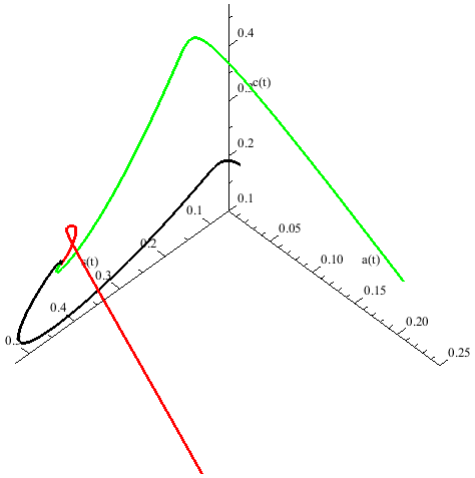

Figure 7. Phase portrait for $R_0 > 1$

Based on the simulation of Figure 4, Figure 5 and Figure 6, show that when the frequency of people using needles together (κ) is increase, the proportion of susceptible individuals decreases, while the proportion of acute infection, chronic carrier and recovered increase in proportion to the value of parameter κ. This means that if the average frequency of people using needles together is bigger, then the infection rate would be greater, which in turn resulted in the growing spread of Hepatitis C.

Based on the simulation of Figure 4, Figure 5, Figure 6 and Figure 7 shows that the proportion fluctuated. This is a result of contact individuals susceptible to hepatitis C patients with acute and chronic. When the proportion of susceptible individuals is dropped, the proportion of acute infection and chronic carrier increases and towards a point. Increasing the proportion of individuals in acute infection and chronic carrier, resulting in increased and the proportion of individuals recovered towards the equilibrium point E_1.

The solution of system (2) given initial values of different moving toward to endemic equilibrium pointE_1 and away from disease free equilibrium E_0. This means that the simulation results with the analytic results in Theorem 4 indicates that when $R_0 > 1$ the endemic equilibrium point E_1 is asymtotically stable and Theorema 2 (ii) which states that when $R_0 > 1$ the disease free equilibrium point E_0 is unstable.

VI. CONCLUSION

Based on the above discussion, it can be concluded that the model form of the spread of hepatitis C in injecting drug users is nonlinear differential equations system in first order. Furthermore, Based on model analysis, resulting the disease free equilibrium point and the endemic equilibrium pointThe spread of hepatitis C virus in injecting drug users model have the basic reproduction number R_0which is an indicator of the spread of disease. If $R_0 < 1$ then the hepatitis C virus does not attack the population or gradually disappeared, whereas if $R_0 > 1$ then the disease is endemic and is very likely to spread. In conditions of $R_0 < 1$, the disease-free equilibrium point is asymptotically stable and when $R_0 \leq 1$ is global asymtotically stable. Meanwhile, when$R_0 > 1$ the disease free equilibrium point is unstable and the endemic equilibrium point is local asymtotically stable.

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