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The Greatest Solution of Inequality A O Kross X Less Than X Less Than B O Dot X By Using A Matrix Residuation Over An Idempotent Semiring

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Abstract— *A complete idempotent semiring has a structure which is called a complete lattice. Because of the same structure as the complete lattice then inequality of the complete idempotent semiring can be solved a solution by using residuation theory. One of the inequality which is explained is* $A \otimes X \leq B$ *where matrices A,X,B with entries in the complete idempotent semiring S. Furthermore, introduced dual product* $\odot$*, i.e. binary operation endowed in a complete idempotent semiring S and not included in the standard definition of complete idempotent semirings. A solution of inequality* $A \otimes X \leq B$ *can be solved by using residuation theory. Because of the guarantee that for each isotone mapping in complete lattice always has a fixed point, then is also exist in the complete idempotent semiring. This of the characteristics is used in order to obtain the greatest solution of inequality* $A \otimes X \leq X \leq B \odot X$ *.*

Keywords: complete lattice, complete idempotent semiring, dual Kleene Star, dual product, residuation theory

I. INTRODUCTION

An idempotent semirings S is a semiring which addition operation $\oplus$ is idempotent. The addition $\oplus$ and multiplication $\otimes$ have neutral elements are denoted ε and e . Idempotent semiring is said to be complete if it is closed for infinite sums and multiplication $\otimes$ distributes over infinite sums.

Because of idempotent property over its sums, idempotent semiring can be endowed ordered relation , denoted $\leq$. A complete idempotent semiring has a same structure with complete lattice. Because of the same structure of its, an inequality of complete idempotent semiring can be had its solution with residuation theory.

Beside of scalar, the concept of residuation can be applied for a equality which is be explain is a inequality $A \otimes X \leq B$ with matrices A, X, B with entries from a complete idempotent semiring S. A greatest solution of the inequality $A \otimes X \leq B$ can be found with residuation theory. First, Baccelli (1992) research how to get the greateast solution of the inequality $A \otimes X \leq B$ with matrices $A \in S^{n \times n}$ and $X \in S^{n \times 1}$. After that, the research is be continued about a greatest solution of the inequality $A \otimes X \leq B$ with matrices $A \in S^{n \times p}$ and $X \in S^{p \times m}$.

As we know, the complete idempotent semiring has a property that its distributes $\otimes$ over infinite sums $\oplus$. Generally, multiplication $\otimes$ distribute over $\wedge$ can't hold, because its just hold that a $\otimes$ (b $\wedge$ c) j (a$\otimes$b)$\wedge$ (a$\otimes$c). The property is distribite multiplication $\otimes$ over $\wedge$ in the complete idempotent semiring can be hold if its has nessesary condisition that element a has a invers, so that a $\otimes$ (b $\wedge$ c) = (a $\otimes$ b) $\wedge$ (a $\otimes$ c). With that the condition, can be defined new operation which has a property of distribute multiplication $\otimes$ over $\wedge$. Hardouin introduce a duality of multiplication $\otimes$ is dual product which is $\odot$ distribute over $\wedge$. Beside of the dual product in the complete idempotent semiring, it is given dual product in idempotent semiring for two matrices $A \in S^{p \times n}$ and $B \in S^{n \times q}$, is denoted $A \odot X$, is defined as $\left(A \odot X\right)_{ij} = \wedge_{k=1,2,\ldots,n} \left(a_{ik} \odot x_{kj}\right)$ with $\wedge$ represents the greatest lower bound..

After that, about a fixed point equation in a complete lattice also can be applied in the complete idempotent semiring. If is given a complete idempotent semiring S and a mapping $f : S \mapsto S$ so its can be collected a nonempty set $\left\{x \in S \middle| f\left(x\right) = x\right\}$. Because of order relation in S, its also can be collected a nonempty set $\left\{x \in S \middle| f\left(x\right) \leq x\right\}$ and $\left\{x \in S \middle| f\left(x\right) \geq x\right\}$. In case of matrix, also can be collected $\left\{X \in S^{n \times m} \middle| f\left(X\right) = X\right\}$, $\left\{X \in S^{n \times m} \middle| f\left(X\right) \leq X\right\}$ and $\left\{X \in S^{n \times m} \middle| f\left(X\right) \geq X\right\}$ with f a isotone mapping. The research before, had be

researched about the solution of inequality $A \otimes X \le B$, is defined the isotone mapping $L_A : X \to A \otimes X$, so that the property of the isotone mapping L_A, can be collected a nonempty set $\left\{ X \in S^{n\times m} \middle| L_A(X) = X \right\}$ and $\left\{ X \in S^{n\times m} \middle| L_A(X) \le X \right\}$.

The research is continued about how to find the least solution of an inequality $A \odot X \ge B$, with is defined a isoton mapping $\wedge_A : X \mapsto A \odot X$, so that the property of the isoton mapping $\wedge_A$ can be collected a nonempty set $\left\{ X \in S^{n\times m} \middle| \wedge_A(X) = X \right\}$ and $\left\{ X \in S^{n\times m} \middle| \wedge_A(X) \ge X \right\}$. With the guarantee of the matrix X which hold $L_A(X) \le X$ and $\wedge_A(X) \ge X$ and also the property $X \ge A \otimes X \Leftrightarrow X \le A \backslash X \Leftrightarrow X = A^* \otimes X \Leftrightarrow X = A^* \backslash X$ dan $X \le B \odot X \Leftrightarrow X \ge B \bullet X \Leftrightarrow X = B_* \odot X \Leftrightarrow X = B_* \bullet X$, so that can research how the characteristic of the solution X which hold a inequality $A \otimes X \le X \le B \odot X$.

II. RESEARCH METHOD

In the research, a researcher has use some step to find a goal. First, its is studied about semiring theory, especially about a definition and a property of complete idempotent semiring. After that, is be researched that there is the relationship between the complete idempotent semiring and complete lattice. Because of the relationship between them, has be understood residuation theory which hold in complete lattice, especially a lower semicontinous mapping and a upper semicontinous mapping, the definition and the property of residual mapping and dual residual mapping. Because of the same structure between complete idempotent semiring and complete lattice, the definition of the lower semicontinous mapping and the definition of the upper semicontinous mapping also hold in the complete idempotent semiring. Beside that, the property of the residuated mapping and the property of the dual residuated mapping also hold in the complete idempotent semiring. So that, residuation theory can be applied for find the greatest solution of the inequality $A \otimes X \le B$ and the least solution of the inequality $A \odot X \ge B$ with matrices A, X, B over complete idempotent semiring.

After that, is defined closure mapping and its characteristics is used for find the least solution of the inequality $A \odot X \ge B$. The research is be continued to find the greatest solution of the inequality $A \otimes X \le B$ with residuation theory. First, is researched the greatest of the inequality $A \otimes X \le B$ with matrices $A \in S^{n\times n}$, $X \in S^{n\times 1}$. The research is be continued the greatest solution of the inequality $A \otimes X \le B$ with matrices $A \in S^{n\times n}$, $X \in S^{n\times n}$ and the solution of the inequality $A \otimes X \le B$ with matrices $A \in S^{n\times p}$, $X \in S^{p\times m}$.

Multiplication is endowed in idempotent semiring, distribute over $\wedge$, so is defined new operation is called dual product in idempotent semiring and dual product in matrix over idempotent semiring. The research is continued about the least solution of the inequality $A \odot X \ge B$, with defined dual residuated mapping

III. MATHEMATICAL BACKGROUND

III.1. RESIDUATION THEORY

Residution has propose for solving unique an equation $f(x) = b$ with inverting isotone mapping f from a complete idempotent semiring D in to another complete idempotent semiring E.

1. f not surjective $\Rightarrow$ $f(x) = b$ will have no solution for some value b.
2. f not injective $\Rightarrow$ $f(x) = b$ will have nonunique subsolution for some value b.

The solution $f(x) = b$ is to consider the subset subsolutions $f(x) = b$, that is value of x satisfying $f(x) \le b$. These following step are used for solving subsolutions $f(x) = b$:

1. This subset of subsolution is nonempty.
2. Choose the upper bounds of the subset.
3. If the upper bounds of the subset exists, it remains to be checked whether the upper bound itself is the subsolution $f(x) = b$. The other words, the subset of subsolutions $f(x) = b$ has a maximum element , is denoted $f^{\#}(b)$, so we can get

$$f^{\#}(b) = \oplus_{\{x|f(x)\le b\}} x \text{ and } f(f^{\#}(b)) \le b$$

4. After we got the maximum element of the subset of subsolutions, we have could be checked whether its maximum element, in this case $f^{\#}(b)$ satisfying an equation $f(x) = b$. If yes, $f^{\#}(b)$ is the solution of the equation $f(x) = b$.

Beside to get the solution $f(x) = b$, for solving $f(x) = b$ can also get supersolutions $f(x) = b$, that is value of x satisfying $f(x) \ge b$. These following step are used for solving supersolutions $f(x) = b$:

1. This subset of supersolution is nonempty, so can be choosed the lower bounds of the subset of supersolutios $f(x) = b$.
2. If the lower bounds of the subset exists, it remains to be checked whether the lower bound itself is the supersolution $f(x) = b$. The other words, the subset of supersolutions $f(x) = b$ has a minimum element , is denoted $f^{b}(b)$, so we can get

$$f^{b}(b) = \oplus_{\{x|f(x)\ge b\}} x \text{ and } f(f^{b}(b)) \ge b$$

3. After we got the minimum element of the subset of supersolutions, we have could be checked whether its minimum element, in this case $f^{b}(b)$ satisfying an equation $f(x) = b$.
4. If yes, $f^{b}(b)$ is the solution of the equation $f(x) = b$.

Definition 3.1. *Let order set D and E with ordered sets $\le$. An isotone mapping f:D → E is said to be* ***residuated*** *if for all $y \in E$, the least upper bound of subset $\{x \in D | f(x) \le y\}$ is exist and is an element of that subset (that maximum element is exist), that maximum element is denoted $f^{\#}(y)$. All of the element x which hold the equality are called subsolutons of the equality f(x) = y.*

Definition 3.2. *Let order set D and E with ordered sets $\le$. An isotone mapping g:D → E is said to be* ***dually residuated*** *if for all $y \in E$, the greatest lower bound of subset $\{x \in D | g(x) \ge y\}$ is exist and is an element of that subset (that minimum element is exist), that minimum element is denoted $f^{b}(y)$. All of the element x which hold the equality are called supersolutons of the equality g(x) = y.*

Theorem 3.3. *If let f be an isotone mapping $f : E \to F$ with E, F are complete idempotent semirings, a bottom element of E is denoted ε_E, a top element of E is denoted T_E, the following statements are equivalent :*

1. *A mapping f is residuated.*
2. *$f(\varepsilon_E) = \varepsilon_F$ and $f(\oplus_{x\in X} x) = \oplus_{x\in X} f(x)$ for all $X \subseteq E$ (that f is lower – semicontinuous).*
3. *$f(\mathrm{T}_E) = \mathrm{T}_F$ and $f^{\#}(\wedge_{y\in Y} y) = \wedge_{y\in Y} f^{\#}(y)$ for all $Y \subseteq F$ (that $f^{\#}$ is upper – semicontinuous).*

Theorem 3.4. *If let g be an isotone mapping $g : E \to F$ with E, F are complete idempotent semirings, a bottom element of F is denoted ε_F, a top element of F is denoted T_F, the following statements are equivalent :*

1. *A mapping g is dually residuated.*
2. *$g(\mathrm{T}_E) = \mathrm{T}_F$ and $g(\wedge_{x\in X} x) = \wedge_{x\in X} g(x)$ for all $X \subseteq E$ (that g is upper – semicontinuous).*
3. *$g(\varepsilon_E) = \varepsilon_F$ and $g^{b}(\oplus_{y\in Y} y) = \oplus_{y\in Y} g^{b}(y)$ for all $Y \subseteq F$ (that g^{b} is lower – semicontinuous).*

III.2. CLOSURE MAPPING

This section, will be explain a closure mapping and that relationship with a residuated mapping and a dually residuated mapping

Definition 3.5. *Let an idempotent semiring S and an isotone mapping* $h : S \mapsto S$ *. If* $h \circ h = h \geq Id_S$ *h is a closure mapping. If* $h \circ h = h \leq Id_S$ *, h is a dual closure.mapping.*

Theorem 3.6 *Let semiring S. If a mapping* $h : S \mapsto S$ *is a residuated mapping, then the following statements are equivalent :*

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1. *h is a closure mapping*
2. $h^{\#}$ *is a dual closure mapping*
3. $h \circ h^{\#} = h^{\#}$
4. $h^{\#} \circ h = h$

IV. THE GREATEST SOLUTION OF AN INEQUALITY $A \otimes X \leq B$ WITH MATRIX A ∈ $S^{N\times P}$ AND B ∈ $S^{N\times M}$

For solving an inequality $A \otimes X \leq B$ with A ∈ $S^{n\times p}$ dan B ∈ $S^{n\times m}$ for using residuation theory. In this case for getting the solution of the inequality $A \otimes X \leq B$ with A ∈ $S^{n\times p}$ and B ∈ $S^{n\times m}$, that is

Given a complete semiring idempotent S, matrix A ∈ $S^{n\times p}$ dan B ∈ $S^{n\times m}$. Defined a mapping L_A $S^{n\times p} \rightarrow S^{n\times m}$, that is

$$L_A : X \mapsto A \otimes X \qquad (1)$$

Generally, the solution of the inequality $A \otimes X \leq B$ for A ∈ $S^{n\times p}$, X ∈ $S^{p\times m}$ and B ∈ $S^{n\times m}$ are $\left[L_A^{\#}(B)\right]_{ik} = \wedge_{j=1}^{n}\left(A_{ji} \backslash b_{jk}\right) = \left[A \backslash B\right]_{ik}$

Because of all B ∈ $S^{n\times m}$, there is exist a maximal element $L_A^{\#}(B)$ yang $A \otimes X \leq B$ then can be create a isotone mapping $L_A^{\#}$: $S^{n\times m} \rightarrow S^{p\times m}$. Furthetmore, a mapping $L_A^{\#}$ is defined for all X ∈ $S^{n\times m}$

$$L_A^{\#} : S^{n\times m} \rightarrow S^{p\times m}$$

$$X \mapsto A \backslash X$$

With $\left[A \backslash X\right]_{ik} = \wedge_{j=1}^{n}\left(A_{ji} \backslash x_{jk}\right)$ is called a residuated mapping L_A with A ∈ $S^{n\times p}$.

V. THE LEAST SOLUTION OF AN INEQUALITY $A \odot X \geq B$ WITH MATRIX A ∈ $S^{N\times P}$ AND B ∈ $S^{N\times M}$

V.1 Dual Product

Distributive multiplication ⊙ over ∧ isn't satisfied in a idempotent semiring, generally, is added necessary condition that the idempotent semiring is semifield. That is motivated definition ⊙ is called dual product.

Definition 5.1. *Let a complete idempotent semiring S. Dual product in S is denoted* ⊙ *is biner operation which is assumed have the following characteristics :*

1. *Operation* ⊙ *is assosiative.*
2. *(S,* ⊙ *) has a netral element e.*
3. *Operation* ⊙ *is distributive over* ∧ *for infinite element, that is u* $\forall a_i \in S, \left(\wedge_{i\in I} a_i\right) \odot b = \wedge_{i\in I}\left(a_i \odot b\right)$
4. *Ttop element* T *is as absorb element over* ⊙ *, that is* $\left(\forall a \in S, \mathrm{T} \odot a = a \odot \mathrm{T} = \mathrm{T}\right)$

Then, will let dual product definition for matrix :

Definition 5.2. *Let a complete idempotent semiring S. Matrix* $A \in S^{n\times p}$,$B \in S^{p\times m}$, *then* $A \odot B$ *is defined:*

$$\left(A \odot B\right)_{ij} = \wedge_{k=1,2,\ldots,p}\left(a_{ik} \odot b_{kj}\right)$$

For all $i = 1, 2, \ldots, n$ *and* $j = 1, 2, \ldots, m$.

V.2. The Least Solution of An Inequality $A \odot X \geq B$ with A ∈ $S^{n\times p}$ and B ∈ $S^{n\times m}$

Theorem 5.3. *Let a complete idempotent semiring S and matrix A* $\in S^{n\times p}$. *A mapping*

$$\wedge_A : S^{p\times m} \mapsto S^{n\times m}$$
$$X \mapsto A \odot X$$

Is a dually residuated mapping and that dual residual can be denoted

$$\wedge_A^b : S^{n\times m} \mapsto S^{p\times m}$$
$$X \mapsto A \bullet X$$

With the definition operation is

$$\left[A \bullet X\right]_{ij} = \oplus_{k=1}^{n} \left(A_{ki} \bullet x_{kj}\right) \qquad (2)$$

$\mathrm{T} \bullet x = \varepsilon, \varepsilon \bullet x = \mathrm{T}$, and $\varepsilon \bullet \varepsilon = \varepsilon$

VI. THE SOLUTION OF INEQUALITY $A \otimes X \leq X \leq B \odot X$

VI. 1 Fixed Point Equations and Inequality of Fixed Point.

Definiton 6.1. *Let a set A and function* $f : A \to A$. *An element* $a \in A$ *is called a* ***fixed point*** *of function* f *if* $f(a) = a$. *Furthermore,* a *can be called a solution of the fixed point equation* $x = f(x)$. *When the set* A *is endowed with a ordered relation* $\leq$, *can be defined* ***prefixed point*** *of* f, *that is* $a \in A$ *is called* ***prefixed points*** *of* f *if* $f(a) \leq a$. *Besides that, can also be defined a post-fixed point of* f, *that is* $a \in A$ *is called* ***post-fixed points*** *of* f *if* $f(a) \geq a$.

Because the complete idempotent semiring S is the ordered set with the ordered relation $\leq$ then it can be defined a fixed point equation of a isoton mapping $f : S \to S$, that is :

$$f(x) = x \qquad (3)$$

and fixed point inequality $f : S \to S$ has the following hold:

$$f(x) \leq x \qquad (4)$$
$$f(x) \geq x \qquad (5)$$

VI.2 Kleene Star

Definition 6.2. *Let a complete idempotent semirings S. Kleene star is a mapping.*

$$k : S \mapsto S, a \mapsto a^* = \oplus_{i=0}^{\infty} a^i$$

With $a^{i+1} = a \otimes a^i$ *and* $a^0 = e$.

Kleene star can also be applied to a square matrix in the complete idempotent semiring.

Definition 6.3. *Let a complete idempotent semiring S. Kleene Star of matrix A* $\in S^{n\times n}$ *is defined by*

$$K : S^{n\times n} \mapsto S^{n\times n}, A \mapsto A^* = \oplus_{i=0}^{\infty} A^i$$

with $A^0 = E$, *E an identity matrix and*

$$A^k = A \otimes A^{k-1}.$$

Propotition 6.4. *Let a complete idempotent semiring S. Matrix A* $\in S^{n\times n}$ *and X*$\in S^{n\times p}$. *A mapping* $L_{A^*} : S^{n\times p} \to S^{n\times p}$, $X \mapsto A^* \otimes X$ *is a closure mapping, so that is,*

$$A^* \otimes A^* \otimes X = A^* \otimes X$$

As a consequence,

$$X = A^* \otimes X \Leftrightarrow X \in \operatorname{Im} L_{A^*} \qquad (3)$$

Lemma 6.5. *Given a complete idempotent semiring S, matrices $A \in S^{n\times n}$ and $X \in S^{n\times p}$*
The following equivalences hold:

1. $X \le A \backslash X$
2. $X \ge A \otimes X$
3. $X = A^* \otimes X$
4. $X = A^* \backslash X$

VI.3. Dual Kleene Star

Definition Kleene star explain that Kleene Star is the sum of infinite $\oplus$ of an element in a complete idempotent semiring. The following duality of Kleene Star i.e. dual Kleene Star which is meet $\wedge$ infinite. of an element in a complete idempotent semiring.

Definition 6.6. *Given a complete idempotent semiring S. Dual Kleene Star is a mapping*

$$l : S \mapsto S, b \mapsto b_* = \wedge_{i=0}^{\infty} b^{\odot i}$$

where $b^{\odot i+1} = b \odot b^{\odot i}$ *and* $b^{\odot 0} = e$

Selanjutnya, pendefinisian dual star dapat pula diaplikasikan dalam kasus ma-trik, sebagaimana dijelaskan mengenai dual star pada matrik.

Definisi 6.7. *Given a complete idempotent semiring S. Dual Kleene Star for matrix $B \in S^{n\times n}$ is defined by*

$$B_* = \wedge_{k=0}^{\infty} B^{\odot k}$$

Where $B^{\odot i+1} = B \odot B^{\odot i}$ *and* $B^{\odot 0} = E$

Sifat 6.8. *Given matrix $B \in S^{n\times n}$. A mapping $\wedge_{B_*}$ is upper semicontinuous mapping and according to Defintioni 5.6., a mapping $\wedge_{B_*} : S^{n\times p} \mapsto S^{n\times p}$, $X \mapsto B_* \odot X$ is a dual closure mapping. So that,*

$$B_* \odot B_* \odot X = B_* \odot X \qquad (4)$$

As consequence is

$$X = B_* \odot X \Leftrightarrow X \in \mathrm{Im} \wedge_{B_*} \qquad (5)$$

Propotition 6.9. *Given a complete idempotent semiring S, matrices $B \in S^{n\times n}$ and $X \in S^{n\times p}$, then the following equivalences hold :*

1. $X \le B \odot X$
2. $X \ge B \bullet X$
3. $X = B_* \bullet X$
4. $X = B_* \odot X$

Proposisi 6.10. *Given a complete idempotent semiring S, matrices A, B $\in S^{n\times n}$ and X $\in S^{n\times m}$. The following equivalences hold :*

$$A \otimes X \le X \le B \odot X \Leftrightarrow X \in \mathrm{Im} L_{A^*} \cap \mathrm{Im} \wedge_{B_*}$$

Proposisi 6.11. *Given a complete idempotent semiring S, and matrices A, B, G $\in S^{n\times n}$. The greatest of solution is X which is hold :*

$$A \otimes X \le X \le B \odot X \text{ dan } X \le G$$

is

$$X = \left(\left(B_* \bullet A^*\right)^*\right) \backslash G$$

Proof :

1. Proved that $A \otimes X \le X \le B \odot X$ dan $X \le G \Rightarrow X \le \hat{X}$. According to Propotition 6.10, $A \otimes X \le X \le B \odot X \Leftrightarrow X \in \mathrm{Im} L_{A^*} \cap \mathrm{Im} \wedge_{B_*}$. That mean matrix X must hold $X = B_* \bullet (A^* \otimes X)$.

$$X = B_* \bullet (A^* \otimes X)$$
$$\Leftrightarrow X = (B_* \bullet A^*) \otimes X)$$
$$\Leftrightarrow X = ((B_* \bullet A^*)^*) \backslash X$$

Therefore, $A \otimes X \le X \le B \odot X$ and $X \le G \Rightarrow X = ((B_* \bullet A^*)^*) \backslash X$ and $X \le G$. Attention that Theorem 6.5, $X = ((B_* \bullet A^*)^*) \backslash X \Leftrightarrow (B_* \bullet A^*) \otimes X \le X$. Bacause of $(B_* \bullet A^*) \otimes X \le X$ and $X \le G$ then $X \le \hat{X} = ((B_* \bullet A^*)^*) \backslash G$.

2. Proved that $\hat{X} \le G, \hat{X} = A^* \otimes \hat{X}$.

First, will proved that $\hat{X} \in \operatorname{Im} L_{A^*}$ is equivalent $\hat{X} = A^* \otimes \hat{X} = A^* \backslash \hat{X}$. According to Lemma 6.5, $\hat{X}$ hold by

$$(B_* \bullet A^*) \otimes \hat{X} \le \hat{X} \le (B_* \bullet A^*) \backslash \hat{X} \qquad (9)$$

Because of an isotone mapping L_{A^*} and $\hat{X} \le (B_* \bullet A^*) \backslash \hat{X}$, then $A^* \otimes \hat{X} \le A^* \otimes ((B_* \bullet A^*) \backslash \hat{X})$. Therefore,

$$A^* \backslash \hat{X} \ge A^* \backslash ((B_* \bullet A^*) \backslash \hat{X})$$

Attention that

$$\begin{aligned} A^* \backslash ((B_* \bullet A^*) \backslash \hat{X}) &= ((B_* \bullet A^*) \otimes A^*) \backslash \hat{X} \\ &= (B_* \bullet (A^* \otimes A^*)) \backslash \hat{X} \\ &= (B_* \bullet A^*) \backslash \hat{X} \end{aligned}$$

So that, be got $A^* \backslash \hat{X} \ge (B_* \bullet A^*) \backslash \hat{X}$. According to inequality 9, $(B_* \bullet A^*) \backslash \hat{X} \ge \hat{X}$. Consequently, get inequality $A^* \backslash \hat{X} \ge (B_* \bullet A^*) \backslash \hat{X} \ge \hat{X}$. Thus $A^* \backslash \hat{X} \ge \hat{X}$. After that, $\hat{X} \ge A^* \backslash \hat{X}$ (because of $A^* \ge E$) then $A^* \backslash \hat{X} = \hat{X}$, i.e. $\hat{X} \in \operatorname{Im} L_{A^*}$.

The second, proved $\hat{X} \in \operatorname{Im} \wedge_{B_*}$, i.e. $\hat{X} = B_* \odot \hat{X} = B_* \bullet \hat{X}$. From the equation 9. Be got $\hat{X} \ge (B_* \bullet A^*) \otimes \hat{X} = B_* \bullet (A^* \otimes \hat{X}) = B_* \bullet \hat{X}$ (because of $\hat{X} = A^* \otimes \hat{X}$). The other side, $B_* \le E^{\odot}$. Because of $\wedge_{B_*}$ an isotone mapping, then $B_* \odot \hat{X} \le E^{\odot} \odot \hat{X}$. Consequently, can be got $\hat{X} \le B_* \bullet \hat{X}$. So that, $\hat{X} = B_* \bullet \hat{X} = B_* \odot \hat{X}$.

Third, because of inequality $(B_* \bullet A^*)^* \ge E$, then $(B_* \bullet A^*)^* \odot G \ge E \odot G$. Consequently, can be got $(B_* \bullet A^*)^* \backslash G \le G$. So that, be got $\hat{X} \le G$.

VII. KESIMPULAN DAN SARAN

VII.1 Conclusion

In this paper, can be concluded that the same structure beetwen a complete idempotent semiring and a complete lattice give more advantages for solving inequality of a complete idempotent semiring by using residuation theory. Given complete idempotent semiring S, then

1. Solution of inequality $A \otimes X \le B$ with matrices $A \in S^{n \times p}$ and $B \in S^{n \times m}$ is $\wedge_{j=1}^{n} (A_{ji} \backslash b_{jk}) = [A \backslash B]_{ik}$.
2. Solution of inequality $A \odot X \ge B$ with matrices $A \in S^{n \times p}$ and $B \in S^{n \times m}$ is $(A \bullet X)_{ij} = \oplus_{k=1}^{n} (A_{ki} \bullet x_{kj})$.
3. Solution of inequality $A \otimes X \le X \le B \odot X$ dan $X \le G$ dengan $A, B, G \in S^{n \times n}$ can be get with formula $\hat{X} = ((B_* \bullet A^*)^*) \backslash G$.

VII.2 Suggestion

1. In case that relation of Propotition 6.10 and Propotition 6.11, is necesarry explanation about the best approximation terbaik an element in G by another element in $\gamma^{b}_{L_{A^*}}$ but that is explained in this paper, yet. So that, can be added for completing the relationship beetwen Proposisi 6.10 and Proposisi 6.11 in this paper.
2. This research can be continued for solving inequality $A \otimes X \leq X \leq B \odot X$ and $X \leq G$, i. e the greatest solution where matrices $A, B, G \in IS^{n \times n}$ with a semiring IS is interval semiring.

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