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The Guarantee of the Existence of Interpolation Functions of Fractional Cubic Spline Using Piecewise Method

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Abstract. The interpolation problem concerns how to construct a function which passess several particular known points. The discourse about interpolation problem has been growing rapidly, regarding both the method and the type of function. One of these developments is to generalize the natural numbers into the fractional numbers as a derivative order. This paper describes how to interpolate the fractional cubic spline function, i.e., third-degree polynomial function that decreased fractionally; then ensures the existence and the uniqueness. Besides knowing the value of function at the points of the partition, the value of the derivative at these points also must be known. Fractional piecewise interpolation was used as the methods, i.e., by partitioning the given interval into sub-intervals. Instead of using the first derivative or second derivative, fractional derivatives were employed in this study. The results showed that the greater number of terms in fractional spline led to the longer proof of the existence and the uniqueness of the interpolant function, the greater the degree spline, and the more complicated of the method of proof.

INTRODUCTION

Interpolation problem is how to build a function through several known focal points. Since Newton in 1675 and Lagrange in 1795 developed the theory and the method of interpolation, in recent decades the interpolation method is growing rapidly, as can be seen elsewhere [1-4]. Furthermore, Unser and Blu in their study [5] discussed about Fractional Spline and Wavelets. On the other hand, Zahra and Elkholy used Cubic Splines as a method in the Numerical Solution of Fractional Differential Equations [6]. Meanwhile, another study developed the Fractional Cubic Spline Interpolations without using the Derivative Values [7].

In addition to the development of methods and development of this type of functionality interpolant, starting in 2000, interpolation theory began to evolve in the direction of application, particularly the issue of minimizing energy. In this case, Wallner in his paper [3] discussed the existence of a curve that minimizes the energy, i.e., the function of a vector in R^n, in which energy-shaped integral derivative function of integer order. Hofer and Pottman also contributed to the development of interpolation that minimizes the problem of energy. They discussed about how the spline interpolation minimizes the energy in a manifold, in which the spline is a piecewise polynomial of degree k that has the function to-derivative (k-1) which is continuous [8]. In another of their study [9], the method on how to minimize the energy of a rigid body motion was presented. Solution functions in the form of interpolant curve was investigated in a k-dimensional regular surface which is located in Euclidean space R^n. Two types of energy wihch are minimized are integral derivative function order-1 and order-2. Similarly, Gunawan *et al* [4] interpolated sine-shaped curve Fourier series which minimized the energy in the form of an integral of the square of the second derivative of a function.

This paper described the spline interpolation in which spline is referred to fractional spline, especially cubic spline with fractional degrees (n , α) and the degree of terms spline decreased by α. Interpolation method which was used is piecewise method, *i.e.*, a method that divides the main interval into some sub-intervals where the function must be smooth and continuous at every partition points. By changing the cubic spline function into a cubic spline interpolation fraction, it is necessary to investigate the existence of the solution of this problem .

INTERPOLATION, METHODS, AND FRACTIONAL CUBIC SPLINE

A broad definition of interpolation is presented in the introduction above. In more detail, the definitions of interpolations are presented as follows:

Definition 1. [1] Let f be a function that is defined on a closed and bounded interval $[a\ ,\ b]$, the values of this function at every partision points x_i are known as $a = x_0 < x_1 < x_2 < \ldots < x_n = b$.
Interpolation of function $f(x)$ by the function $u(x)$ means to construct a function $u(x)$ such that

$$u(x_i) = f(x_i)\ ,\quad i = 0, 1, 2, 3, \ldots, n.$$

In addition to the method of Lagrange that rely on the conditions $u(x_i) = f(x_i)$ for all i, and Hermit methods which also requires $u'(x_i) = f'(x_i)$, one of the another known classic interpolation method is the method piecewise, where interpolan function $u(x)$ in $[a\ ,\ b]$ is divided into $u_i(x)$ according to subinterval $[x_{i-1}\ ,\ x_i]$. Therefore

$$u(x) = u_i(x)\quad ;\ i = 1, 2, 3, \ldots, n.$$

Further, understanding generally of the spline is a piecewise polynomial of degree k, where the derivative function with $(k-1)$ order is continuous. [1]. The following definitions are given for the spline.

Definition 2.[10] Let $f(x)$ be a function that is defined on a closed and bounded interval $[a\ ,\ b]$. Also let $a = x_0 < x_1 < x_2 < \ldots \le x_n = b$ are partition points of $[a\ ,\ b]$. Spline or Interpolan Spline is a piecewise polynomial

$$u(x) = u_i(x)\ ,\ x_{i-1} \le x \le x_i \quad ;\ i = 1, 2, \ldots, n$$

that satisfies the following properties:

1. $u_i(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$,
   which means that the value of the function at any partision points is the same.
2. $u_i(x_i) = u_{i+1}(x_i)$ for $i = 1, \ldots, n-1$, which means that spline continuous at each x_i .
3. $u_i'(x_i) = u_{i+1}'(x_i)$ for $i = 1, \ldots, n-1$, which means that spline forms a smooth curve.
4. $u_i''(x_i) = u_{i+1}''(x_i)$ for $i = 1, \ldots, n-1$, which means that the second derivative is continuous at $[a\ ,\ b]$.

One of the following boundary conditions apply, namely:

a. $u''(a) = u''(b) = 0$, that called natural boundary conditions
b. $u'(a) = f'(a)$ dan $u'(b) = f'(b)$, that called clambed boundary conditions.

If $u(x)$ satisfies the natural boundary conditions, we say that $u(x)$ is a natural spline. [10]

In case piecewise polynomial has degree $n = 3$, then called Cubic Spline. In this paper, the main subject is a Fractional Cubic Spline, ie the spline of three degrees that degree of terms spline decreased by fractionally. Fractional Cubic Spline definition in general is given below.

Definitions 3. Spline fractional around x_i degree $(n\ ,\ \alpha)$ with n natural numbers and rational numbers α is

$$u(x) = \sum_{k=0}^{n} a_{n-k}\,(x - x_i)^{(n-k)\alpha}$$

$$= a_n(x - x_i)^{n\alpha} + a_{n-1}(x - x_i)^{(n-1)\alpha} + a_{n-2}\,(x - x_i)^{(n-2)\alpha} + \cdots + a_1\,(x - x_i)^{\alpha} + a_0$$

where a_i and k is a real constant.
For example, form of fractional spline with degree $(n\ ,\ \alpha) = (3\ ,\ 1/3)$ is

$$u(x) = a_3(x - x_i) + a_2(x - x_i)^{2/3} + a_1(x - x_i)^{1/3} + a_0$$

that satisfies the following conditions:

$$u_i^{\left(\frac{2}{3}\right)}(x_i) = f_i^{\left(\frac{2}{3}\right)}(x_i)\ ,\ u_i^{\left(\frac{1}{3}\right)}(x_i) = f_i^{\left(\frac{1}{3}\right)}(x_i)\quad ,i = 1, 2, \ldots, n\ ,\ \text{dan}\ \ u(x_0) = f(x_0)\ ,$$

where $f_i^{\left(\frac{1}{3}\right)}(x_i)$ is fractional derivative of function f_i with order $\frac{1}{3}$ at the point x_i .

The second example, for $n = 4$ and $\alpha = 0{,}5$, fractional spline with degree $(n\ ,\ \alpha)$ is

$$u(x) = a_4(x - x_i)^2 + a_3(x - x_i)^{1.5} + a_2\,(x - x_i) + a_1\,(x - x_i)^{0.5} + a_0$$

that satisfies the following conditions:

$$u_i^{(1.5)}(x_i) = f_i^{(1.5)}(x_i)\quad ,\ u_i^{(0.5)}(x_i) = f_i^{(0.5)}(x_i)\ ,\ \text{dan}\ \ u(x_0) = f(x_0)\ .$$

In this paper, the main subject centered around fractional cubic spline, ie the fractional spline with degree $n = 3$. Thus degrees $(n\ ,\ \alpha)$ of Fractional Cubic Spline can be obtained from $n = 6$ andt $\alpha = 0.5$, or $n = 9$ and $\alpha = 0.3$, or some other equivalent value . So the general form of Fractional Cubic Spline with degree (6, 0.5) are

$$u(x) = a_6(x-x_i)^3 + a_5(x-x_i)^{2.5} + a_4\,(x-x_i)^2 + a_3\,(x-x_i)^{1.5} + a_2\,(x-x_i) + a_1(x-x_i)^{0.5} + a_0 \qquad (1)$$

that satisfies conditions

$$u_i^{(2.5)}(x_i) = f_i^{(2.5)}(x_i) \;\;,\; u_i^{(1.5)}(x_i) = f_i^{(1.5)}(x_i)\,,\; u_i^{(0.5)}(x_i) = f_i^{(0.5)}(x_i) \;\text{ and }\; u(x_0) = f(x_0)\,,$$

where the fractional derivative of function x^p with order α is

$$D_x^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p+1-\alpha)} x^{p-\alpha}. \qquad [11]$$

THE EXISTENCE AND THE UNIQUENESS FRACTIONAL SPLINE INTERPOLATION

This section discussed two theorems which are the focussess of this study, *i.e.*, the existence and the uniqueness of simple fractional spline interpolation and the existence and the uniqueness of fractional cubic spline interpolation.

The existence and the uniqueness spline with degree (2 , α)

Before presenting the problem of existence of fractional cubic spline interpolation, firstly, the existence of fractional spline interpolation with more simple degree $(2,\alpha)$, will be discussed to see the pattern of more modest evidence before it is adopted to prove a higher degree case.

Theorem 1: Let $u^{(0.5)}(x) \in C^2[0,1]$, and on each subinterval $[x_i\,,\,x_{i+1}]$ interpolan function $u(x)$ has degree $(2,\alpha)$. If given the real numbers $f(x_0)$, and $u(x_i) = f(x_i)$ for $i = 0, 1, \ldots, n$, then there exist spline $u(x)$ that unique such that

$$u_i^{(0.5)}(x_i) = f_i^{(0.5)}(x_i) \;\;,\; i = 1, 2, \ldots, n \;\text{ and }\; u(x_0) = f(x_0) \qquad (2)$$

Spline fractional that satisfies (2) on $[x_i\,,\,x_{i-1}]$ has form

$$u(x) = u_i\,A_0(t) + u_{i+1}\,A_1(t) + h^{1/2}u_i^{(0.5)}\,A_2(t) \qquad (3)$$

where

$$A_0(t) = 1 - t^{3/2} \;\;,\; A_1(t) = t^{3/2} \;\;,\; A_2(t) = \frac{2}{\sqrt{\pi}}\left(t^{\frac{1}{2}} - t^{\frac{3}{2}}\right) \qquad (4)$$

and $x = x_i + th$, $t \in [0\,,\,1]$.

Coefficient u_i in (3) is expressed in a recursive formula:

$$u_i = u_{i-1} - \frac{2}{3\sqrt{\pi}}\,h^{\frac{1}{2}}\left(f_{i-1}^{(0.5)} - 2\,f_i^{(0,5)}\right)\;,\; u(x_0) = f(x_0) \qquad (5)$$

Proof:

$p(t)$ on $[0\,,\,1]$ can be expressed in the form

$$p(t) = p_0\,A_0(t) + p_1\,A_1(t) + p_0^{(0.5)}\,A_2(t).$$

To determine A_0 , A_1 , dan A_2 , the above equation we can write for $p(t) = 1\,, t^{\frac{1}{2}}, t^{\frac{3}{2}}$ resulting in

$$A_0 + A_1 = 1 \;\;;\;\; A_1 + \frac{\sqrt{\pi}}{2}A_2 = t^{\frac{1}{2}} \;\;;\;\; A_1 = t^{\frac{3}{2}}\;.$$

Thus obtained (6).

Furthermore, to a certain i where $i = 0\,,\,1, \ldots, n$, the set $x = x_i + th$, $t \in [0\,,\,1]$. On subinterval $[x_i\,,\,x_{i-1}]$ fractional spline $u(x)$ that satisfies (4), is:

$$u(x) = u_i\,A_0(t) + u_{i+1}\,A_1(t) + h^{1/2}u_i^{(0.5)}\,A_2(t)$$

Form of this equation will be the same on subinterval $[x_{i-1}\,,\,x_i]$.
From continuity requirements $u^{(0,5)}(x_i^-) = u^{(0.5)}(x_i^+)$ then obtained a recurrence equation (5).
So the theorem has been proven.

The existence and the uniqueness spline with degree (6 , α): spline cubic fractional

Let $u^{(0.5)}(x) \in C^{6}[0,1]$, and on each subinterval $[x_i , x_{i-1}]$ intepolant function $u(x)$ has formed:

$$u(x) = a_6(x-x_i)^3 + a_5(x-x_i)^{2.5} + a_4(x-x_i)^2 + a_3(x-x_i)^{1.5} + a_2(x-x_i) + a_1(x-x_i)^{0.5} + a_0 \quad (6)$$

where a_i is the real constant.

Theorem 2: Suppose $u(x)$ is the fractional cubic spline intepolan in form (6). If given the real numbers: $u(x_i) = f(x_i)$ and $f(x_0)$, then $u(x)$ is unique and satisfies:

$$u_i^{(0.5)}(x_i) = f_i^{(0.5)}(x_i) \,,\; u_i^{(1.5)}(x_i) = f_i^{(0.5)}(x_i) \,,\; u_i^{(2.5)}(x_i) = f_i^{(2.5)}(x_i) \quad (7)$$

and $u(x_0) = f(x_0)$ for $i = 0, 1 , \ldots, n$.

Fractional cubic spline that satisfies (7) on $[x_i , x_{i-1}]$ has form:

$$u(x) = u_i\, A_0(t) + u_{i+1}\, A_1(t) + h^{1/2}\left[f_i^{(0.5)}\, A_2(t) + f_{i+1}^{(0.5)}\, A_3(t)\right] + h^{3/2}\left[f_i^{(1.5)}\, A_4(t) + f_{i+1}^{(1.5)}\, A_5(t)\right] + h^{5/2}\, f_i^{(2.5)}\, A_6(t) \quad (8)$$

where

$$A_0(t) = \frac{1}{3}\left(176\, t^{\frac{9}{2}} - 99\, t^{\frac{7}{2}} - 80\, t^{\frac{11}{2}} + 3\right) + 1$$

$$A_1(t) = \frac{1}{3}\, t^{\frac{7}{2}}(80\, t^2 - 176\, t - 99)$$

$$A_2(t) = -\frac{2}{63\sqrt{\pi}}\, t^{\frac{1}{2}}\,(656\, t^5 - 1520\, t^4 + 927\, t^3 - 63)$$

$$A_3(t) = -\frac{256}{63\sqrt{\pi}}\, t^{\frac{7}{2}}\,(8\, t - 9)(t-1)$$

$$A_4(t) = -\frac{4}{945\sqrt{\pi}}\, t^{\frac{3}{2}}\,(1360\, t^4 - 337 t^3 + 2331\, t^2 - 315)$$

$$A_5(t) = \frac{256}{945\sqrt{\pi}}\, t^{\frac{7}{2}}\,(10\, t - 9)(t-1)$$

$$A_6(t) = -\frac{8}{945\sqrt{\pi}}\, t^{\frac{5}{2}}\,(4\, t - 3)(20\, t - 21)(t-1)$$

and $x = x_i + th$, $t \in [0 , 1]$. With the same statement for $u(x)$ on $[x_{i-1} , x_i]$.
Coefficient u_i in (8) expressed in a recursive formula:

$$\frac{1155}{16}\sqrt{\pi}\,(s_i - s_{i-1}) = h^{1/2}\left[\frac{355}{8}\, f_{i-1}^{(0,5)} + 100 f_i^{(0.5)}\right] + h^{3/2}\left[\frac{115}{12}\, f_i^{(1.5)} - \frac{40}{3} f_{i-1}^{(1.5)}\right] + h^{5/2}\left[f_{i-1}^{(2.5)} + \frac{59}{12} f_i^{(2.5)}\right]$$

and $u(x_0) = f(x_0)$. [1]

Proof:
In this case we can express every $p(t)$ on$[0 , 1]$ in the following form:

$$p(t) = p_0\, A_0(t) + p_1\, A_1(t) + p_0^{(0.5)}\, A_2(t) + p_1^{(0.5)}\, A_3(t) + p_0^{(11.5)}\, A_4(t) + p_1^{(1.5)}\, A_5(t) + p_0^{(22.5)}\, A_6(t)$$

And for determine coefficient A_j , $j = 0 , 1 , \ldots , 6$ the above equation can be written for:

$$p(t) = 1 , t^{\frac{1}{2}} , t^{\frac{3}{2}} , t^{\frac{5}{2}} , t^{\frac{7}{2}} , t^{\frac{9}{2}} , t^{\frac{11}{2}} .$$

By the same method as in theorem 1, the desired results are obtained.
Thus the theorem is proved.

CONCLUSION

Based on what has been discussed above, firstly, it can be concluded that the spline of degree n can be generalized into a fractional spline of degree (n , α), in which the first term had degree $n\alpha$ and degree of the next terms reduced by α. Thus, regardless of the value α, the numbers of terms in fractional spline of degree (n , α) is $(n + 1)$, equal to the numbers of terms of polinomial or spline of degree n which is ordinary. Secondly, the initial condition of piecewise interpolation methods that usually contains the first derivative and second derivative also

can be generalized into a fractional derivatives, *i.e.*, derivatives with fractional order. The results showed that the greater number of terms in fractional spline led to the longer proof of the existence and the uniqueness of the interpolant function, the greater the degree spline, and the more complicated of the method of proof. In this paper, the authors only proved the theorem for fractional cubic spline with degrees (6 , 0.5).

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